The Jacobi weighted ray transform

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Based on joint work with Lauri Oksanen

Motivation

Let (M, g) be a smooth compact Lorentzian (or Riemannian) manifold with a smooth boundary. Let

$$\mathcal{A} = \{u \in C^\infty(M) \,:\, \Delta_g u = 0 \quad \text{on} \quad M^{\mathrm{int}}\}.$$

We are interested in "density" properties for products of two or more elements of the set $\mathcal{A}$. Precisely, we fix $m \geq 2$ and ask whether

$$\int_M q(x)\, u_1(x) u_2(x) \dots u_m(x)\, dV_g = 0 \quad \text{for all } u_1, \dots, u_m \in \mathcal{A}$$

for some $q \in C(M)$ implies that $q \equiv 0$ on M?

We will focus on the case $m = 3$.

The case $m = 2$ on Lorentzian manifolds

Question. Does $\int_M q\, u_1 u_2 \, dV_g = 0$ for all $u_1, u_2 \in \mathcal{A}$ imply that $q \equiv 0$?

$$\mathcal{A} = \{u \in C^\infty(M) \,:\, \Delta_g u = 0 \quad \text{on } M^{\mathrm{int}}\}.$$

This question has applications in Calderón type inverse problems for linear equations. In the Lorentzian case, the density problem reduces to studying injectivity of the *light ray transform of q:*

$$\mathcal{L}_\gamma(q) = \int_I q(\gamma(s))\, ds, \quad \gamma \text{ a null geodesic.}$$

Injectivity of $\mathcal{L}$ is known in the following cases:

- *M is a real-analytic manifold, g is real-analytic and (M, g) satisfies a convex foliation property* [Stefanov'18].
- *(M, g) is a stationary Lorentzian manifold and sastisfies a certain convexity condition,* [A.F-Ilmavirta-Oksanen'20].

The case $m = 2$ on Euclidean domains

Question. Does $\int_\Omega q\, u_1 u_2\, dx = 0$ for all $u_1, u_2 \in \mathcal{A}$ imply that $q \equiv 0$?

$$\mathcal{A} = \{u \in C^\infty(\overline{\Omega}) \,:\, \Delta u = 0 \quad \text{on } \Omega\}.$$

For Euclidean domains, completeness is well understood:

- [Calderón'80] proves completeness on $\mathcal{A}$, using complex geometric optics, i.e
$$u = e^{i\zeta\cdot x}, \quad \text{where} \quad \zeta\cdot\zeta = 0.$$
- [Dos Santos Ferreira-Kenig-Sjöstrand-Uhlmann'09] proves completeness on
$$\mathcal{B} = \{u \in C^\infty(\Omega) \,:\, \Delta u = 0 \quad \text{and} \quad u|_\Gamma = 0\},$$
where $\Gamma \subset \partial\Omega$ is an arbitrary proper closed subset.

The case $m = 2$ on CTA manifolds

For general Riemannian manifolds, all of the results are stated for conformally transversally anistropic manifolds (CTA):

$$M \Subset \mathbb{R} \times M_0 \quad \text{and} \quad g(t,x) = c(t,x)(dt^2 \oplus g_0(x)).$$

- If $c \equiv 1$, (M_0, g_0) is real-analytic and satisfies an additional assumption, then completeness is known [Krupchyk-Liimatainen-Salo'20].

In general the density problem reduces to question of injectivity of the *geodesic ray transform* on (M_0, g_0) [Ferreira-Kurylev-Lassas-Salo'13]:

$$\mathcal{I}_\gamma(f) = \int_I f(\gamma(s))\, ds,$$

where $\gamma : I \to M_0$ is an inextendible unit speed geodesic on M_0.

Injectivity of $\mathcal{I}$ is known when

- (M_0, g_0) is "simple" [Mukhometov'77]...
- (M_0, g_0) has a strictly convex boundary and has a foliation by strictly convex hypersurfaces. [Uhlmann-Vasy'15],...

The case $m \geq 4$

Consider the Cauchy data set

$$\mathscr{C}(V) = \{(u, \partial_\nu u)|_{\partial M} \,:\, u \in C^\infty(M) \quad \text{and} \quad \Delta_g u + V(x,u) = 0 \quad \text{on } M^{\text{int}}\},$$

where

$$V(x,z) = \sum_{k=2}^{\infty} V_k(x) z^k, \quad \text{with} \quad V_k \in C^\infty(M).$$

Question. Does $\mathscr{C}(V)$ uniquely determine V?

Using multiple-fold linearization [Kurylev-Lassas-Uhlmann'14] this question can be reduced to density properties for products of solutions to $\Delta_g u = 0$. Previous results include:

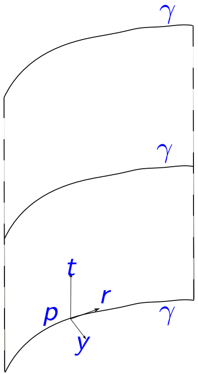
- Lorenztian case: [Kurylev-Lassas-Uhlmann'18],[Lassas-Uhlmann-Wang'16],[A.F-Oksanen'19],[Hintz-Uhlmann-Zhai'20]...
- Riemannian case on CTA manifolds: [Lassas-Liimatainen-Lin-Salo'19],[A.F-Oksanen'19]...

Gaussian quasi-modes in CTA manifolds

Let

$$M \Subset \mathbb{R} \times M_0 \quad \text{and} \quad g(t,x) = c(t,x)(dt^2 \oplus g_0(x)).$$

For simplicity we assume that $c \equiv 1$. Let $\gamma : I \to M_0$ be a unit speed geodesic and consider the Fermi coordinates (r, y) near $\gamma(r) = (r, 0)$. Gaussian quasimode solutions "concentrate" on the planes $\mathbb{R} \times \gamma$ [Dos Santos Ferreira-Kurylev-Lassas-Salo'13].



Gaussian quasi-modes in CTA manifolds

There are two families of Gaussian quasimode solutions to $\Delta_g u = 0$ of the form

$$U_\lambda = e^{\lambda t}\left(e^{i\lambda r + \frac{i}{2}\lambda H(r)y\cdot y + \dots}((\det Y(r))^{-\frac{1}{2}} + \dots)\chi(y) + R_\lambda\right),$$

and

$$\tilde{U}_\lambda = e^{-\bar{\lambda} t}\left(e^{-i\bar{\lambda} r - \frac{i}{2}\bar{\lambda}\bar{H}(r)y\cdot y + \dots}((\det \bar{Y}(r))^{-\frac{1}{2}} + \dots)\chi(y) + \tilde{R}_\lambda\right),$$

where $\lambda = \tau + i\sigma$, $H = \dot{Y}Y^{-1}$ and Y is a (1,1)-tensor that solves the Jacobi equation (e.g [Dahl'08], [Katchalov-Kurylev-Lassas])

$$\ddot{Y} - KY = 0 \quad \text{K is the } (1,1) \text{ - Ricci curvature tensor}$$

in the $(n-2)$-dimensional orthogonal complement $\dot{\gamma}^\perp$ of $\dot{\gamma}$, together with the additional constraint

$$\dot{Y}(r_0)Y^{-1}(r_0) \text{ is symmetric}, \quad \Im(\dot{Y}(r_0)Y^{-1}(r_0)) > 0 \quad \text{for some } r_0.$$

A simple fact from linear algebra

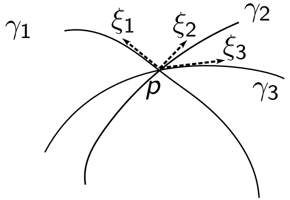
Given a point $p \in M_0$ and unit speed geodesics γ_1, γ_2, γ_3 passing through p, the Gaussian quasimode $U^{(j)}_{\lambda_j}$ and $\tilde{U}^{(j)}_{\lambda_j}$ take the form

$$U^{(j)}_{\lambda_j} \approx e^{\tau_j t + i\tau_j \xi_j \cdot x} a_{\lambda_j} \quad \text{and} \quad \tilde{U}^{(j)}_{\lambda_j} \approx e^{-\tau_j t + i\tau_j \xi_j \cdot x} \tilde{a}_{\lambda_j}$$

near $\mathbb{R} \times p$ where $\tau_j = \Re \lambda_j$ and the vectors $\xi_j \in T_p M_0$ satisfy

$$g_0(\xi_j, \xi_j) = 1 \quad \text{for } j = 1, 2, 3.$$

The leading parts of the phases must cancel out for the product of the three Gaussian quasi modes. This implies that $\xi_2, \xi_3 \in \mathrm{Span}\,(\xi_1)$.



$m = 3$: reduction to an integral transform

We consider the integral

$$0 = \int_M q\, U_\lambda^2\, \tilde{U}_{2\lambda}\, dV_g$$

that has the following principal part:

$$\int_M q\, e^{4i\sigma t - 4\sigma r - 2\tau \Im H y \cdot y + \dots}\, (|\det Y|^{-1} (\det Y)^{-\frac{1}{2}} + \dots) \chi^3(y)\, dt\, dr\, dy.$$

Recall that $|\det \Im H|^{\frac{1}{2}} = c |\det Y|^{-1}$ for some $c > 0$. Applying the method of stationary phase gives

$$\int_{\mathbb{R}} \hat{q}(-4\sigma, r, 0) e^{-4\sigma r} (\det Y(r))^{-\frac{1}{2}}\, dr,$$

where q is extended by zero outside of M and $\hat{q}$ denotes the Fourier transform of q with respect to t. This leads to the inversion of an integral transform along γ in M_0.

Jacobi transform on Riemannian manifolds (M, g)

We denote by $\mathbb{Y}_\gamma$, the set of complex Jacobi (1,1)-tensors along a geodesic γ on M that are normal to $\dot{\gamma}$ and additionally satisfy the condition

(C) $\dot{Y}(r_0)Y^{-1}(r_0)$ is symmetric and $\Im(\dot{Y}(r_0)Y^{-1}(r_0)) > 0$ for some r_0.

The Jacobi transform is then defined as follows:

$$\mathcal{J}_\gamma f(Y) = \int_\gamma f(r)(\det Y(r))^{-\frac{1}{2}}\, dr$$

where $f \in C(\mathbb{R})$ is supported on $\gamma^{-1}(M)$.

Theorem [A.F- Oksanen] Assume that $\dim M = 2, 3$. Let $\gamma : I \to M$ be an inextendible geodesic on M with no conjugate points. Then $\mathcal{J}_\gamma$ is injective, that is to say,

$$\int_\gamma f(r)(\det Y(r))^{-\frac{1}{2}}\, dr = 0 \quad \forall Y \in \mathbb{Y}_\gamma \implies f \equiv 0.$$

Inversion in Euclidean geometry

In Euclidean space $(\mathbb{R}^n, \mathbb{E}^n)$ Jacobi fields are affine along a geodesic segment $\gamma = (r, 0)$ with $r \in I$. Indeed,

$$\ddot{Y}(r) = 0 \quad \text{for } r \in I.$$

We can assume without loss of generality that $0 \notin I$. Next, given $\epsilon > 0$ we choose the following Jacobi matrix Y_ϵ of size $(n-1) \times (n-1)$:

$$Y_\epsilon(r) = \begin{pmatrix} r - i\epsilon & 0 & \cdots & 0 \\ 0 & r - i\epsilon & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r - i\epsilon \end{pmatrix}$$

Note that Y_ϵ satisfies condition (C) for all $\epsilon > 0$. Since $\mathcal{J}_\gamma f(Y_\epsilon) = 0$ for all $\epsilon > 0$, we obtain that

$$\int_I f(r)(r - i\epsilon)^{-\frac{n-1}{2}}\, dr = 0, \quad \text{for all } \epsilon > 0.$$

Inversion in Euclidean geometry continued

Expanding in Taylor series of ϵ, we deduce that

$$\int_I f(r) r^{-\frac{n-1}{2}} r^{-k}\, dr = 0, \quad \text{for all } k = 0, 1, 2, \ldots$$

As the set $\{r^{-k}\}_{k=0}^{\infty}$ is dense in $C(I)$, we conclude that

$$f \equiv 0.$$

Inversion of the Jacobi transform in Euclidean spaces is a key ingredient of the following result:

Theorem [CÂRSTEA-A.F] Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary. The set

$$\mathrm{Span}\{\nabla u_1 \otimes \nabla u_2 \otimes \ldots \otimes \nabla u_m \,:\, u_1, \ldots, u_m \text{ harmonic in } \bar{\Omega}\}$$

with $m \geq 3$ is dense in $C(\bar{\Omega}; \mathbb{C}^{\otimes k})$.

Inversion of the Jacobi transform when dim $M = 2$

We suppose that $\mathcal{J}_\gamma f = 0$ and want to show that $f = 0$. Let $\hat{M}$ be an extension of M and choose $\gamma(a)$ to be a point outside M such that no point on γ is conjugate to $\gamma(a)$. Now, consider the normal Jacobi fields, $Y_k(r)$ with $k = 1, 2$ satisfying $\ddot{Y}_k - KY_k = 0$

$$Y_1(a) = 0, \quad \dot{Y}_1(a) = 1 \quad \text{and} \quad Y_2(a) = 1, \quad \dot{Y}_2(a) = 0.$$

Let

$$Y_\epsilon = Y_1 - i\epsilon Y_2$$

for $\epsilon > 0$ and observe that the condition (C) holds. By the non-conjugacy assumption imposed on γ, $Y_1(r) > 0$ on supp$f$ and

$$0 = \int_{\mathbb{R}} f(r)Y(r)^{-\frac{1}{2}}\,dr = \int_{\mathbb{R}} \tilde{f}(r)(1 - \epsilon X(r))^{-\frac{1}{2}}\,dr$$

where $\tilde{f} = fY_1^{-\frac{1}{2}}$ and $X = Y_2 Y_1^{-1}$.

Inversion of the Jacobi transform continued

By expanding in Taylor series in ϵ, we deduce that

$$\int_{\mathbb{R}} \tilde{f}(r)X(r)^k \, dr = 0 \quad \text{for } k = 0, 1, 2, \ldots.$$

Supposing that we can change the variable $s = X(r)$, we deduce that

$$\int_{\mathbb{R}} h(s)s^k \, ds = 0 \quad \text{for } k = 0, 1, 2, \ldots$$

where $h(s) = \tilde{f}(r(s))\dot{X}(r(s))$. This implies that $h = 0$ and subsequently that $f = 0$.
To justify the change of variables, observe that since $X(r) = Y_2(r)Y_1^{-1}(r)$:

$$\dot{X}(r) = W(r)Y_1^{-2}(r), \quad W(r) = \dot{Y}_2(r)Y_1(r) - Y_2(r)\dot{Y}_1(r)$$

where the Wronskian W satisfies $W(r) = W(a) = -1$.

Inversion of the Jacobi transform for dim $M = 3$

We suppose that

$$\mathcal{J}_\gamma f = \int_I f(r)(\det Y)^{-\frac{1}{2}}\, dr = 0 \quad \forall\, Y \in \mathbb{Y}_\gamma$$

and want to show that if $p \in \gamma$, then $f(\gamma^{-1}(p)) = 0$. We start by writing $p = \gamma(0)$ and define for $\epsilon > 0$:

$$Y_\epsilon(r) = X(r) - i\epsilon Z(r), \quad \forall\, r \in I,$$

where $X(r)$ and $Z(r)$ are real-valued (1,1)-Jacobi matrices on $\dot{\gamma}^\perp(r)$ subject to

$$X(0) = 0, \quad \dot{X}(0) = Id. \quad \text{and} \quad Z(0) = Id, \quad \dot{Z}(0) = 0.$$

Since X is of rank two and since no points on γ are conjugate to p it follows that

$$\det X(r) > 0 \quad \forall r \in I \setminus \{0\}.$$

Inversion of the Jacobi transform for dim $M = 3$ continued

Note that

$$\int_I f(r)\Im(\det(X - i\epsilon Z))^{-\frac{1}{2}}\, dr = 0 \forall\, \epsilon > 0.$$

Since $\det X > 0$ away from $r = 0$ it follows that

$$|\Im(\det(X - i\epsilon Z))^{-\frac{1}{2}}| \leq C\epsilon \quad \text{away from } r = 0.$$

On the other hand near the point $r = 0$, we have that

$$\Im(\det(X - i\epsilon Z))^{-\frac{1}{2}} \approx \epsilon(t^2 + \epsilon^2)^{-1}.$$

and since $\int_I \epsilon(r^2 + \epsilon^2)^{-1}\, dr \to \pi$ as $\epsilon \to 0$, we can conclude that

$$f(0) = 0.$$

Thank You.