Light ray transform and inverse problems in cosmology

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Inverse problems in cosmology

The existence of gravitational waves was predicted by Einstein and confirmed by the LIGO project in 2015. The gravitational waves generated in the early Universe, called **primordial gravitational waves**, are of great interest in cosmology. The detection of these gravitational waves is quite challenging:

> "*· · · will involve waves today whose wave lengths will extend all the way up to our present cosmological horizon (the distance out to which we can currently observe in principle) and that are likely to be well beyond the reach of any direct detectors for the foreseeable future.* "

quoted from Krauss, Dodelson and Meyer in *Science*, 2010.

Inverse problems in cosmology

As demonstrated by Sachs and Wolfe 1967, primordial fluctuations create Cosmic Microwave Background (CMB) anisotropies, and the redshift measurement is connected to a cosmological X-ray transform of the gravitational fluctuations.

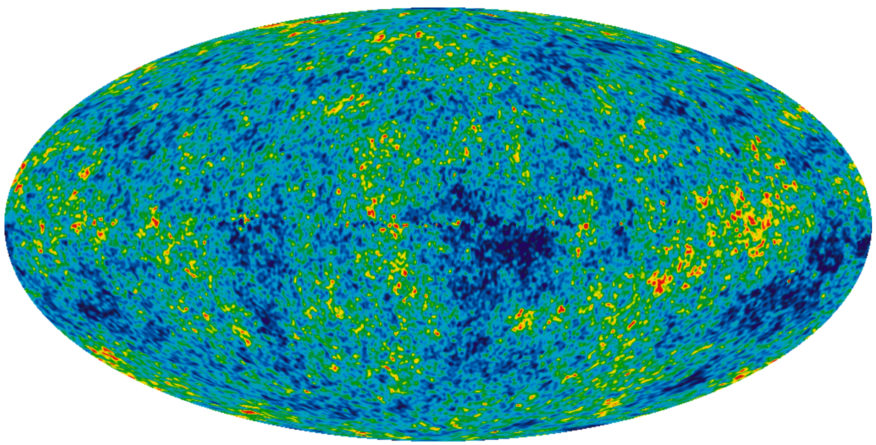

Figure: All-sky picture of the infant universe created from Wilkinson Microwave Anisotropy Probe (WMAP) data. Picture courtesy to NASA.

The **inverse problem** we study is the determination of early gravitational perturbations from the anisotropies of CMB measurements.

Cosmological X-ray tomography

The mathematical problem can be formulated as follows. Let $(\mathscr{M}, g_0)$ be a Friedman-Lemaître-Robertson-Walker (FLRW) cosmological model, where

$$\mathscr{M} = (0, \infty) \times \mathbb{R}^3, \quad g_0(x) = -dt^2 + R^2(t)dy^2$$

and $x = (t, y), t \in \mathbb{R}_+, y \in \mathbb{R}^3$, and $R(t) > 0$ is smooth. In this model, the Universe starts from a Big Bang at $t = 0$ and inflates. The factor R reflects the rate of expansion.

- when the Universe was very young and dominated by radiation, the factor $R(t) \approx t^{\frac{1}{2}}$.
- At later times, when matter became to dominate, $R(t) \approx t^{2/3}$.
- Based on more recent observations, the Universe is expanding with a rate $R(t) = e^{\Lambda t}$ with Λ a positive cosmological constant.

Cosmological X-ray tomography

Consider a perturbation $(\mathscr{M}, g)$ of $(\mathscr{M}, g_0)$.

- let $\mathscr{S}_0 = \{t_0\} \times \mathbb{R}^3$ be the "surface of last scattering" which is the moment that photons start to travel freely in space-time.
- let $\mathscr{S} = \{t_1\} \times \mathbb{R}^3$ be the surface where we observe the CMB.
- let $\gamma(\tau), \tau \geq 0$ be a light-like geodesic on $(\mathscr{M}, g)$ with $\gamma(0) \in \mathscr{S}_0$. We think of $\gamma(\tau)$ as the trajectory of photons emitted from $\mathscr{S}_0$.

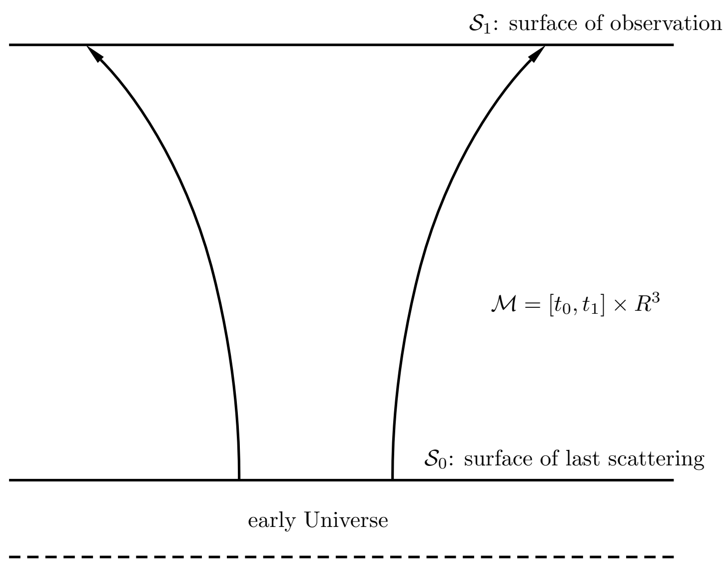

Figure: The FLRW cosmology model.

Cosmological X-ray tomography

The initial energy of the photon observed by an observer at $\mathscr{S}_0$ moving in tangent direction $\beta = \partial_t$ is $E_0 = g(\dot\gamma(0), \beta)$. The energy received by the observer at $\mathscr{S}$ is $E = g(\dot\gamma(\tau_0), \beta)$, where τ_0 is such that $\gamma(\tau_0) \in \mathscr{S}$. The **redshift** is defined as

$$\mathcal{R} = \frac{E_0 - E}{E}.$$

The linearization of $\mathcal{R}$ leads to a tomography problem. Suppose that the metric perturbation g is a one parameter family

$$g_\epsilon = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \cdots .$$

Cosmological X-ray tomography

Let $\mathcal{R}_\epsilon$ be the redshift measurement for g_ϵ and consider the linearization $\partial_\epsilon \mathcal{R}_\epsilon|_{\epsilon=0}$. Then

$$\partial_\epsilon \mathcal{R}_\epsilon|_{\epsilon=0} = \frac{1}{2R(t_0)} X_{g_0}(R^2 \mathcal{L}_{R\beta}(R^{-2} g_1)) \qquad (1)$$

where $\mathcal{L}_\bullet$ denotes the Lie derivative for a vector field $\bullet$.

For a light-like geodesic $\gamma(\tau), \tau \in \mathbb{R}$ on $(\mathscr{M}, g_0)$, we define the **cosmological X-ray (or light-ray) transform** of a symmetric two tensor f by

$$X_{g_0}(f)(\gamma) = \int_{\mathbb{R}} \sum_{i,j=0}^{3} f_{ij}(\gamma(\tau))\dot{\gamma}^i(\tau)\dot{\gamma}^j(\tau)d\tau.$$

Cosmological X-ray tomography

(1) is essentially the **Sachs-Wolfe effects** in the primordial perturbation problem. For the derivation, see Sachs and Wolfe 1967, Lassas, Oksanen, Stefanov and Uhlmann 2018.

Let's consider the following problem for a moment.

Problem (tomography problem)

Determine g_1 *from the transform* (1).

For example, g_ϵ could represent interesting astrophysical objects such as cosmic strings, domain walls, see Damour and Vilenkin 2000.

Cosmological X-ray tomography

- Guillemin 1989 investigated the transform on the compactification of $\mathbb{R}^{2+1}$ in the study of the Lorentzian version of the Zoll problem. There are limitations on the ability for reconstruction, see Greenleaf and Uhlmann 1990.
- For Minkowski space-time $\mathbb{R}^{3+1}$, the transform is microlocally invertible in space-like directions, fails to be invertible in time-like directions, and invertibility is unclear near the light-like directions. See LOSU 2018 and Wang 2018. The result in LOSU has been generalized to Lorentzian manifolds by the same authors in 2020.
- The Minkowski light ray transform is injective for C_0^∞.
  - Stefanov 2017 : support theorems for analytic Lorentzian manifolds.
  - Ilmavirta 2018 : Pestov identity method.
  - Feizmohammadi, Ilmavirta and Oksanen 2020: injectivity for certain static and stationary spacetimes.
  - Krishnan, Senapati and Vashisth 2020: uniqueness result for tensors.

The inverse Sachs-Wolfe problem

For the primordial gravitational wave problem, we should take into account that g_ϵ satisfies the Einstein equations with certain source fields and initial perturbations at $\mathscr{S}_0$ from g_0.

On the linearization level, this implies that the perturbation g_1 satisfies some wave equations. These are known in cosmology literatures such as Mukhanov, Feldman and Brandenberger 1992, Dodelson 2003, Durrer 2008 ect.

For ease of elaboration, we consider the scalar type perturbation.

The inverse Sachs-Wolfe problem

Let's assume that the actual cosmos is a metric perturbation $g = g_0 + \delta g$ on $\mathscr{M}$ where δg is a small perturbation compared to g_0.

We use the conformal time s such that $ds = R^{-1}dt$. Then we get $g_0 = R^2(s)(ds^2 - \delta_{ij}dx^i dx^j) = R^2(s)g_M$ where g_M is the Minkowski metric on $M = (0, \infty)$. In the longitudinal gauge, also called the conformal Newtonian gauge, we consider the metric g of the form

$$g = R^2(s)[(1 + 2\Phi)ds^2 - (1 - 2\Psi)dx^2] \tag{2}$$

Here, Φ, Ψ are scalar functions on M and this type of perturbation is called **scalar perturbations**.

The inverse Sachs-Wolfe problem

Let T be the temperature observed at $\mathscr{S}$ in the isotropic background g_0. Let δT be the temperature fluctuation from the isotropic background. Then one component of $\delta T/T$ is the **integrated Sachs-Wolfe (ISW) effects**

$$(\frac{\delta T}{T})^{ISW} = \int_0^{s_1 - s_0} (\partial_s \Phi(\gamma(\tau)) + \partial_s \Psi(\gamma(\tau)) d\tau \tag{3}$$

see e.g. Durrer 2008. This quantity depends on the light ray γ which indicates the anisotropy. The integrated Sachs-Wolfe effect can be extracted from the CMB and other astrophysical measurements, see for example Manzotti, Dodelson 2014.

The inverse Sachs-Wolfe problem

Consider Universe dominated by perfect fluid sources. In the case of adiabatic perturbations, one can derive that $\Phi = \Psi$ and Φ satisfies the following Bardeen's equation

$$\Phi'' + 3H(1 + c_s^2)\Phi' + c_s^2 \Delta\Phi + [2H' + (1 + 3c_s^2)H^2]\Phi = 0, \qquad (4)$$

where $H(s) = \partial_s R(s)/R(s)$. In general, the right hand side of the equation is a non-zero term related to the entropy perturbations. Prescribing Cauchy data of Φ at $\mathscr{S}_0$, one can solve the Cauchy problem of (4) to get Φ in M.

Problem (inverse Sachs-Wolfe problem)

Determining Φ from (3) where Φ satisfies the Cauchy problem of (4).

The main results

Our main results are about the **stable determination** of solutions of Cauchy problem from the light ray transform.

- Let $M = [t_0, t_1] \times \mathbb{R}^3$ and $t_0 = 0$. Let $g_M = -dt^2 + dx^2$ be the Minkowski metric on M. Let X_M be the light ray transform.
- Let $c > 0$ be a constant. Denote $\Box_c = \partial_t^2 + c^2\Delta$ where Δ is the positive Laplacian on $\mathbb{R}^3$. Here, c is the wave speed. Consider the Cauchy problem

$$\begin{aligned} \Box_c f = 0 \quad \text{on } M \\ f = f_1, \ \ \partial_t f = f_2, \quad \text{on } \mathscr{S}_0. \end{aligned} \tag{5}$$

The main results

Let $\mathcal{N}^s \stackrel{\text{def}}{=} H^{s+1}_{\text{comp}}(\mathscr{S}_0) \times H^s_{\text{comp}}(\mathscr{S}_0)$.

Theorem (Vasy-Wang, 2019)

Suppose $0 < c \leq 1$ *is constant. Assume that* $(f_1, f_2) \in \mathcal{N}^s$, $s \geq 0$, *and* f_1, f_2 *are supported in a compact set* $\mathscr{K}$ *of* $\mathscr{S}_0$. *Then* $X_M f$ *uniquely determines* f *and* f_1, f_2 *which satisfy* (5). *Moreover, there exists* $C > 0$ *such that*

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \leq C\|X_M f\|_{H^{s+2}(\mathscr{C})}$$
$$\text{and } \|f\|_{H^{s+1}(M)} \leq C\|X_M f\|_{H^{s+2}(\mathscr{C})}$$

where $\mathscr{C}$ *is the set of light rays on* M.

The main results

We also prove the results

- When the wave operator is of the form

$$P(x,t,\partial_x,\partial_t) = \Box_c + P_1(x,t,\partial_x,\partial_t) + P_0(x,t)$$

  where P_1 is a first order differential operator with real valued smooth coefficients and P_0 is smooth.

- For small metric perturbations $g_\delta = g_M + h$ with $h = \sum_{i,j=0}^{3} h_{ij} dx^i dx^j$ where
  (A1) h is a symmetric two tensor smooth on M;
  (A2) for $\delta > 0$ small, the seminorm $\|h_{ij}\|_{C^3} = \sup_{(t,x)\in M} \sum_{|\alpha|\leq 3} |\partial^\alpha h_{ij}(t,x)| < \delta$, $i,j = 0,1,2,3$.

Other interesting problems

- **Source problem** for the wave equation

$$\Box u = f, \text{ on } M$$
$$u = 0 \text{ for } t < t_0$$

where f is compactly supported in M.

- **Metric perturbation**, on-going work with A. Vasy. The model is the Einstein-Euler system

$$\mathrm{Ric}_{\mu\nu}(g) - \Lambda g_{\mu\nu} = G_g T_{\mu\nu}^{scalar}$$
$$\nabla_\alpha(\sigma^s g^{\alpha\beta}\nabla_\beta\Phi) = 0$$

Other interesting problems

- **Kinetic equation:** consider source problem for the linear Boltzmann equation or non-stationary transport equation on $\mathbb{R}^n$:

$$\partial_t u(t,x,\theta) + \theta \cdot \nabla_x u(t,x,\theta) + \sigma(t,x,\theta)u(t,x,\theta)$$
$$= \int_{\mathbb{S}^{n-1}} k(t,x,\theta,\theta')u(t,x,\theta')d\theta' + f(t,x),$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \theta \in \mathbb{S}^{n-1}$. Here, σ is the absorption coefficient, k is the scattering kernel and f is the source term. Consider the inverse problem of determining the source term f from the measurement of u at $t = T > 0$.

Proofs: the light ray transform (I)

The idea of the proofs is the following. For the Cauchy problem, we can represent the solution as

$$u = E_1 f_1 + E_2 f_2$$

Then we have

$$X_M u = X_M E_1 f_1 + X_M E_2 f_2$$

We will study the operator $X_M E_i, i = 1, 2$ and modify it to a pseudo-differential operator on $\mathbb{R}^3$.

Proofs: the light ray transform (I)

We parametrize the set of light rays $\mathscr{C}$ as follows: let $x_0 \in \mathscr{S}_0$ and $v \in \mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$. Then a light ray from x_0 in direction $(1, v)$ is

$$\gamma(\tau) = (t_0, x_0) + \tau(1, v), \tau \in [0, t_1 - t_0].$$

In particular, we can identify $\mathscr{C} = \mathbb{R}^3 \times \mathbb{S}^2$. The light ray transform for scalar functions on (M, g_M) is defined by

$$X_M(f)(\gamma) = \int_0^{t_1 - t_0} f(\gamma(\tau))d\tau, \quad f \in C_0^\infty(M). \tag{6}$$

Proofs: the light ray transform (I)

Consider the Cauchy problem

$$\begin{aligned} \Box_c u = 0, \quad &\text{on } M^\circ = (t_0, t_1) \times \mathbb{R}^3 \\ u = f_1, \quad \partial_t u = f_2, \ &\text{on } \mathscr{S}_0 = \{t_0\} \times \mathbb{R}^3. \end{aligned} \tag{7}$$

The fundamental solution can be written down quite explicitly. For general strictly hyperbolic equations, Duistermaat-Hörmander constructed a parametrix for the Cauchy problem. One can find a parametrix when the equation contains lower order terms.

Proofs: the light ray transform (I)

In this case,

$$u(t, x) = E_+ h_1 + E_- h_2,$$

where that $E_\pm$ are represented by oscillatory integrals

$$E_\pm(f)(t, x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i((x-y)\cdot\xi \pm ct|\xi|)} f(y) dy d\xi.$$

and

$$\hat{h}_1 = \frac{1}{2}(\hat{f}_1 + \frac{1}{ic|\xi|}\hat{f}_2), \quad \hat{h}_2 = \frac{1}{2}(\hat{f}_1 - \frac{1}{ic|\xi|}\hat{f}_2).$$

Proofs: the light ray transform (I)

For $c = 1$, the singularities of the solutions of (7) are all in light-like directions. Consider the composition $X_M \circ E_\pm$.

Let φ be a smooth function on $\mathbb{S}^2$, and I^φ be the integration operator on $C^\infty(\mathbb{R}^3 \times \mathbb{S}^2)$ defined by

$$I^\varphi f(y) = \int_{\mathbb{S}^2} \varphi(v) f(y, v) dv.$$

Then we consider the composition $I^\varphi \circ X_M \circ E_\pm$ as an operator from $C^\infty(\mathscr{S}_0)$ to $C^\infty(\mathscr{S}_0)$.

Proofs: the light ray transform (I)

For technical reasons, we introduce a smooth cut-off function. For $\epsilon > 0$ small, let $\chi_\epsilon(t)$ be a smooth cut-off function on $\mathbb{R}$ such that $\chi_\epsilon(t) = 1$ for $2\epsilon < t < t_1 - 2\epsilon$ and $\chi_\epsilon(t) = 0$ for $t < \epsilon$ and $t > t_1 - \epsilon$. The key step is

Proposition

$K_\pm \doteq I^\varphi X_M \chi_\epsilon E_\pm \in \Psi^{-1}(\mathscr{S}_0)$ *are pseudo-differential operators of order* -1

Proofs: the light ray transform (I)

Consider the oscillatory integral integral representation of the Schwartz kernel K_+

$$K_+(y,z) = (2\pi)^{-6} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{t_1} \int_{\mathbb{R}^3} e^{i((y-x)\cdot\eta + tv\cdot\eta + (x-z)\cdot\xi + t|\xi|)} \varphi(v)\chi_\epsilon(t)\, d\xi\, dt\, dx\, d\eta\, dv$$

We integrate in x, η to get

$$K_+(y,z) = (2\pi)^{-3} \int_{\mathbb{S}^2} \int_0^{t_1} \int_{\mathbb{R}^3} e^{i(y\cdot\xi + tv\cdot\xi - z\cdot\xi + t|\xi|)} \varphi(v)\chi_\epsilon(t)\, d\xi\, dt\, dv$$

Proofs: the light ray transform (I)

Consider the integral in v. For t non-zero, the v integral is non-degenerate with stationary points at $v = \pm\xi/|\xi|$. Applying stationary phase argument, we get

$$K_+(y,z) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i(y-z)\cdot\xi} k_+(\xi) d\xi$$

where

$$k_+(\xi) = +2\pi i|\xi|^{-1}\varphi(-\xi/|\xi|) \int_0^{t_1} t^{-1}(1+\varphi^-(t,\xi))\chi_\epsilon(t)dt$$
$$-2\pi i|\xi|^{-1}\varphi(\xi/|\xi|) \int_0^{t_1} e^{2it|\xi|} t^{-1}(1+\varphi^+(t,\xi))\chi_\epsilon(t)dt$$

The argument works for characteristic functions but additional FIOs arise.

Proofs: the light ray transform (I)

We write

$$X_M \chi_\epsilon f = X_M \chi_\epsilon E_+ h_1 + X_M \chi_\epsilon E_- h_2.$$

Let φ be a smooth function on $\mathbb{S}^2$. Applying I^φ we get

$$I^\varphi X_M \chi_\epsilon f = I^\varphi X_M \chi_\epsilon E_+ h_1 + I^\varphi X_M \chi_\epsilon E_- h_2 = K^{\varphi,+} h_1 + K^{\varphi,-} h_2$$

where we added φ to the notation of $K_\pm$ to emphasize the dependency. Finally, we choose different φ and find operators A_1, A_2 such that

$$A_1 X_M \chi_\epsilon f = h_1 + R_1 h_1 + R_1' h_2, \quad A_2 X_M \chi_\epsilon f = h_2 + R_2 h_1 + R_2' h_2$$

Proofs: the light ray transform (II)

For the second approach, we look at the normal operator

$$E^* L^* L E, \text{ where } E = E_\pm$$

and L is the light ray transform on $\mathbb{R}^{n+1}$.

There are issues about the composition as it stands, and we need to fine tune the operator so it becomes a pseudo-differential operator on $\mathscr{S}_0$. Moreover, we will show that the principal symbol is non-vanishing so the operator can be microlocally inverted.

Proofs: the light ray transform (II)

Our idea is to modify the normal operator $N = L^*L$.

- We let $\chi_{[t_0,t_1]}$ be the characteristic function for $[t_0, t_1]$ on $\mathbb{R}$.
- Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} \chi \subset (t_1, T)$.

Note that $\chi_{[t_0,t_1]}\chi = 0$. Then we consider the composition $E^*\chi L^* L\chi_{[t_0,t_1]}E$. The necessity of the cut-off function χ is demonstrated in the next result. Let $\mathscr{N} = (0, T) \times \mathbb{R}^n$.

Lemma

The composition $\chi L^* L\chi_{[t_0,t_1]}E_\pm \in I^{-n/2+1/4}(\mathscr{N}, \mathscr{S}_0; C_{wv}^\pm)$ are elliptic Fourier integral operators.

Proofs: the light ray transform (II)

To understand the mechanism behind the composition, we use the Lagragian distribution point of view.

For the normal operator $N = L^* L$, it is known see Stefanov and Uhlmann book in progress that the Schwartz kernel of N is

$$K_N(t, x, t', x') = \int_{\mathbb{R}^{n+1}} e^{i(t-t')\tau + i(x-x')\cdot\xi} k(\tau, \xi) d\tau d\xi$$

with

$$k(\tau, \xi) = C_n \frac{(|\xi|^2 - \tau^2)_+^{\frac{n-3}{2}}}{|\xi|^{n-2}}, \quad C_n = 2\pi|\mathbb{S}^{n-2}|.$$

So N is a pseudo-differential operator with a singular symbol. We can show it is a paired Lagrangian distribution introduced in Melrose and Uhlmann 1979, Guillemin and Uhlmann 1981.

Proofs: the light ray transform (II)

Consider two Lagrangians

$$\begin{aligned}\Lambda_0 = \{(t, x, \tau, \xi; t', x', \tau', \xi') \in T^*\mathbb{R}^{n+1}\backslash 0 \times T^*\mathbb{R}^{n+1}\backslash 0 :\\ t = t', x = x', \tau = -\tau', \xi = -\xi'\}\end{aligned} \tag{8}$$

which is the punctured conormal bundle of the diagonal in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and

$$\begin{aligned}\Lambda_1 = \{(t, x, \tau, \xi; t', x', \tau', \xi') \in T^*\mathbb{R}^{n+1}\backslash 0 \times T^*\mathbb{R}^{n+1}\backslash 0 :\\ x = x' + (t - t')\xi/|\xi|, \tau = \pm|\xi|, \tau' = -\tau, \xi' = -\xi\},\end{aligned} \tag{9}$$

which is the flow out of Λ_0 under the Hamilton vector field H_f with $f(\tau, \xi) = \frac{1}{2}(\tau^2 - |\xi|^2)$.

Theorem (Wang 2018)

For the Minkowski light ray transform L, the Schwartz kernel of the normal operator $N = L^*L$ belongs to $I^{-n/2,n/2-1}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}; \Lambda_0, \Lambda_1)$, in which Λ_0, Λ_1 are two cleanly intersection Lagrangians defined in (8), (9)*. The principal symbols of N on $\Lambda_0\backslash\Sigma, \Lambda_1\backslash\Sigma$ are non-vanishing.*

Proofs: the light ray transform (II)

We look at $E^*\chi N\chi_{[t_0,t_1]}E$, with $E = E_k^{\pm}, k = 1, 2.$

1. As $\chi \cdot \chi_{[t_0,t_1]} = 0$, we know that $\chi N\chi_{[t_0,t_1]} \in I^{-n/2}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}; \Lambda_1)$ at least when the characteristic function $\chi_{[t_0,t_1]}$ were smooth. Note that the role of χ is to keep the kernel of N away from the diagonal Λ_0!
2. We will show that Λ_1 intersect $\Lambda_\pm$ cleanly with excess one so the composition $\chi N\chi_{[t_0,t_1]}E \in I^*(\mathscr{N}, \mathscr{S}; C_{wv})$ as a result of Duistermaat-Guillemin's clean FIO calculus with the order $*$ to be determined. For this, we need to address some issue caused by the characteristic function.
3. We can compose with E^* by using clean FIO calculus again to conclude that $E^*\chi N\chi_{[t_0,t_1]}E \in \Psi^*(\mathscr{S})$.

Thank you!