A Nonlinear Plancherel Theorem, Global Well-Posedness for the Defocusing Davey-Stewartson Equation and the Inverse Boundary Value Problem of Calderón in Dimension 2

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Joint work with Idan Regev and Daniel I. Tataru (2020)

IZIPS August 2021

Overview

The Davey-Stewartson Equations

The Davey-Stewartson family of equations were initially introduced in the study of water waves (they model the evolution of weakly nonlinear surface water waves in 2+1 dimensions, travelling principally in one direction).

They also arise in the context of ferromagnetism, plasma physics, and nonlinear optics, and have been shown to have a certain universal character: a large class of nonlinear dispersive equations reduce to the Davey Stewartson system in the limit of weak nonlinearity.

LWP for the L^2 critical case and GWP for small initial data have been proved for various subclasses of this family using dispersive PDE methods: Ghidaglia and Saut (1990), Linares and Ponce (1993), Hayashi and Saut (1995) . High precision numerics: Klein, McLaughlin and Stoilov (2019).

In this talk we consider one special member of this family: defocusing DSII.

The Defocusing DSII Equations

Defocusing DSII:

$$\begin{cases} i\partial_t q + 2(\bar{\partial}^2 + \partial^2)q + q(g + \overline{g}) = 0 \\ \bar{\partial} g + \partial(|q|^2) = 0 \\ q(0, z) = q_0(z). \end{cases} \tag{1}$$

This model is completely integrable and can be solved by the Inverse-Scattering method.

Notation:

$$z = x_1 + ix_2; \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

- Perry (2014) - GWP for general $q_0 \in H^{1,1}$ using Inverse-Scattering method
- This talk: GWP for q_0 in L^2 (mass critical case), via a Plancherel Theorem for the Scattering Transform.

The Scattering Transform

Lax pair for defocusing DSII: $L_t = [L, A]$, where

$$L : \begin{cases} \bar{\partial} m^1 & = q m^2 \\ (\partial + ik) m^2 & = \overline{q} m^1 \end{cases} \tag{2}$$

and

$$A = \ldots \tag{3}$$

Solve (2) with $m^1(z,k) \to 1, m^2(z,k) \to 0$ as $|z| \to \infty$. Define the Scattering Transform:

$$\mathbf{s}(k) := \mathcal{S}q(k) = -\frac{i}{\pi} \int_{\mathbb{R}^2} e_k(z) \overline{q(z)} m^1(z,k) dz. \tag{4}$$

where $e_k(z) = e^{i(zk + \overline{zk})}$ and $dz = dx_1 dx_2$. Then

$$\frac{\partial}{\partial t} \mathbf{s}(t,k) = 2i(k^2 + \overline{k}^2) \mathbf{s}(t,k). \tag{5}$$

Using the Scattering Transform

Similar $\bar{\partial}$ equations in the k variable, so the Inverse-Scattering Transform turns out to be:

$$\mathcal{I}\mathbf{s}(z) = -\frac{i}{\pi}\int_{\mathbb{R}^2} e_z(k)\overline{\mathbf{s}(k)}m^1(z,k)dk. \tag{6}$$

Can we solve the Cauchy problem for defocusing DSII with initial data in L^2 by the following procedure ?

$$\begin{cases} \mathbf{s}_0(k) & = \mathcal{S}q_0(k) \\ \mathbf{s}(t,k) & = e^{2i(k^2+\bar{k}^2)t}\mathbf{s}_0(k) \\ q(t,z) & = \mathcal{I}\big(\mathbf{s}(t,k)\big)(z). \end{cases} \tag{7}$$

$$\begin{array}{ccc} q_0(z) & \xrightarrow{\text{nonlin}} & q(t,z) \\ \Big\downarrow \mathcal{S} & & \mathcal{I} \Big\uparrow \\ \mathbf{s}_0(k) & \xrightarrow{\text{linear}} & \mathbf{s}(t,k). \end{array}$$

Nonlinear Plancherel Identity

This approach to DSII was first introduced by Ablowitz and Fokas (1982, 1984). Beals and Coifman (1995,1988) proved that for q in Schwartz class $\mathbf{s}$ is in Schwartz class and the whole procedure is rigorous. Moreover they showed:

$$\int_{\mathbb{R}^2} |\mathbf{s}(k)|^2 dk = \int_{\mathbb{R}^2} |q(z)|^2 dz.$$

Open Problem: true for all q in L^2 ?

- L.Y.Sung (1994) - q in $L^2 \cap L^p$ for some $p \in [1,2)$ with $\hat{q}$ in $L^1 \cap L^\infty$
- R. Brown (2001) - q in L^2 with small norm
- A. Tamasan (2004) if $q \in W^{\varepsilon,p}$, $p > 2$, then $\mathbf{s} \in L^r$ for all $r > 2/(\varepsilon + 1)$
- P. Perry (2014) - q in weighted Sobolev space $H^{1,1}$
- K. Astala, D. Faraco and K. Rogers (2015) - q in weighted Sobolev space $H^{\varepsilon,\varepsilon}$, $\varepsilon > 0$
- R. Brown, K. Ott and P. Perry (2016) - $q \in H^{\alpha,\beta}$ iff $\mathbf{s} \in H^{\beta,\alpha}$, $\alpha, \beta > 0$.

Plancherel Theorem

Theorem (N.-Regev-Tataru)

The nonlinear scattering transform $\mathcal{S} : q \mapsto \mathbf{s}$ is a C^1 diffeomorphism $\mathcal{S} : L^2 \to L^2$, satisfying:

1. *The Plancherel identity:* $\|\mathcal{S}q\|_{L^2} = \|q\|_{L^2}$
2. *The pointwise bound:* $|\mathcal{S}q(k)| \leq C(\|q\|_{L^2}) M\hat{q}(k)$ *for a.e.* k
3. *Locally uniform bi-Lipschitz continuity:*

$$\frac{1}{C}\|\mathcal{S}q_1 - \mathcal{S}q_2\|_{L^2} \leq \|q_1 - q_2\|_{L^2} \leq C\|\mathcal{S}q_1 - \mathcal{S}q_2\|_{L^2}$$

   with $C = C(\|q_1\|_{L^2})C(\|q_2\|_{L^2})$.

4. *Inversion Theorem:* $\mathcal{S}^{-1} = \mathcal{S}$.
5. *Symplectomorphism property:*

$$\omega_2\left(\frac{\delta\mathcal{S}}{\delta q}\Big|_q q_1, \frac{\delta\mathcal{S}}{\delta q}\Big|_q q_2\right) = \omega_1(q_1, q_2).$$

A bit about the Proof

Making the substitution

$$m_{\pm} = m^1 \pm e_{-k} m^2,$$

we need to solve

$$\begin{cases} \frac{\partial}{\partial \bar{z}} m_{\pm} = \pm e_{-k} q \overline{m_{\pm}} \\ m_{\pm} \to 1 \text{ as } |z| \to \infty. \end{cases}$$

In integral form,

$$m_{\pm} - 1 = (\bar{\partial} \mp e_{-k} q \bar{\cdot})^{-1} \bar{\partial}^{-1} (e_{-k} q).$$

1. For $q \in L^2$, we need new bounds on $\bar{\partial}^{-1}(e_{-k} q)$ which allow us to capture the large k decay without assuming any smoothness on q.
2. We need bounds on $(\bar{\partial} \mp e_{-k} q \bar{\cdot})^{-1}$ which depend only on the L^2 norm of q.

New Estimate on Fractional Integrals

Lemma

For $q \in L^2(\mathbb{C})$,

$$\|\bar{\partial}^{-1}(e_{-k} q)\|_{L^4} \lesssim \|q\|_{L^2}^{\frac{1}{2}} \Big(M\hat{q}(k) \Big)^{\frac{1}{2}}.$$

M is the Hardy-Littlewood Maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

which yields a bounded operator on L^p for $1 < p \leq \infty$.

Theorem (N.-Regev-Tataru)

For $0 < \alpha < n$, $f \in L^p(\mathbb{R}^n)$, $1 < p \leq 2$

$$\left| (-\Delta)^{-\frac{\alpha}{2}} f(x) \right| \leq c_{n,\alpha} \Big(M\hat{f}(0) \Big)^{\frac{\alpha}{n}} \Big(Mf(x) \Big)^{1-\frac{\alpha}{n}}$$

Sketch of Proof - Fractional Integrals

Proof.

Using Littlewood-Paley decomposition,

$$(-\Delta)^{-\frac{\alpha}{2}} f(x) = \frac{1}{(2\pi)^n} \sum_{j=-\infty}^{j_0} \int_{\mathbb{R}^n} \psi_j(\xi) \frac{e^{ix\cdot\xi}}{|\xi|^\alpha} \hat{f}(\xi) d\xi + \sum_{j_0+1}^{\infty} \dots$$

with $\psi_j(\xi) = \psi(\xi/2^j)$ supported in $2^{j-1} < |\xi| < 2^{j+1}$. For $j \le j_0$ use

$$\int_{|\xi|<r} |\hat{f}(\xi)| d\xi \le c_n r^n M\hat{f}(0)$$

...

$$\left|(-\Delta)^{-\frac{\alpha}{2}} f(x)\right| \lesssim 2^{j_0(n-\alpha)} M\hat{f}(0) + 2^{-j_0\alpha} Mf(x)$$

optimize over j_0. □

Key Theorem - bounds in terms of $\|q\|_{L^2}$

Theorem (N.-Regev-Tataru)

Let $q \in L^2$. Then for each $f \in \dot{H}^{-\frac{1}{2}}$ there exists a unique solution $u \in \dot{H}^{\frac{1}{2}}$ of

$$L_q u := \bar{\partial} u + q\bar{u} = f \tag{8}$$

with

$$\|u\|_{\dot{H}^{\frac{1}{2}}} \leq C(\|q\|_{L^2})\|f\|_{\dot{H}^{-\frac{1}{2}}}. \tag{9}$$

In particular, for $f \in L^{\frac{4}{3}}$ the same holds, with $\|u\|_{L^4} \leq C(\|q\|_{L^2})\|f\|_{L^{\frac{4}{3}}}$.

The proof is based on a concentration compactness/profile decomposition argument. Novelty: the Kenig and Merle induction on energy is applied exclusively on the elliptic static problem.

Construction of the Jost Solutions for $q \in L^2$

As a result of the new estimates on fractional integrals and the Key Theorem, we can now establish

Theorem (Jost Solutions)

Suppose $q \in L^2$, then for almost every k there exist unique Jost solutions $m_\pm(z,k)$ with $m_\pm(\cdot,k) - 1 \in L^4$ and moreover

$$\|m(\cdot,k)_\pm - 1\|_{L^4} \le C(\|q\|_{L^2})\big(M\hat{q}(k)\big)^{\frac{1}{2}}$$

$$\|m_\pm - 1\|_{L^4_z L^4_k} \le C(\|q\|_{L^2}).$$

$$\|\bar{\partial} m^1(\cdot,k)\|_{L^{\frac{4}{3}}} \le C(\|q\|_{L^2})\big(M\hat{q}(k)\big)^{\frac{1}{2}}.$$

Scattering Transform as a ΨDO

Recall

$$\mathcal{S}q(k) = \hat{\overline{q}}(k) - \frac{i}{\pi}\int_{\mathbb{R}^2} e_k(z)\overline{q(z)}(m^1(z,k)-1)dz.$$

Replace $\overline{q}$ by the Fourier transform of some function in L^2. Then the above becomes a pseudo-differential operator with symbol $m^1 - 1$. We'd like to prove it is a bounded operator on L^2.

Theorem (N.-Regev-Tataru)

Let $0 \le \alpha < n$. Suppose $a(x,\xi)$ satisfies

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |a(x,\xi)|^{\frac{2n}{n-\alpha}} dx d\xi < \infty \qquad \text{and} \qquad \|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\xi)\|_{L_\xi^{\frac{2n}{n+\alpha}}} \in L_x^{\frac{2n}{n-\alpha}}.$$

Then the pseudo-differential operator

$$a(x,D)f(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi) d\xi \tag{10}$$

is bounded on L^2. Moreover, we have the pointwise bound

$$|a(x,D)f(x)| \le c_{\alpha,n}(Mf(x))^{\alpha/n}\|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\cdot)\|_{L^{\frac{2n}{n+\alpha}}}\|f\|_{L^2}^{1-\frac{\alpha}{n}} \tag{11}$$

for a.e. x.

This completes the sketch of the proof of the Plancherel Theorem.

GWP for Defocusing DSII on L^2

Theorem (N.-Regev-Tataru)

Given $q_0 \in L^2$, there exists a unique solution to the Cauchy Problem for defocusing DSII such that:

1. *Regularity:*

$$q(t,z) \in C(\mathbb{R}, L^2_z(\mathbb{C})) \cap L^4_{t,z}(\mathbb{R}\times\mathbb{C}).$$

2. *Uniform bounds:* $\|q(t,\cdot)\|_{L^2} = \|q_0\|_{L^2}$ *for all* $t \in \mathbb{R}$ *and*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |q(t,z)|^4 dzdt \leq C(\|q_0\|_{L^2}).$$

3. *Stability: if* $q_1(t,\cdot)$ *and* $q_2(t,\cdot)$ *are two solutions corresponding to initial data* $q_1(0,\cdot)$ *and* $q_2(0,\cdot)$ *with* $\|q_j(0,\cdot)\|_{L^2} \leq R$ *then*

$$\|q_1(t,\cdot) - q_2(t,\cdot)\|_{L^2} \leq C(R)\|q_1(0,\cdot) - q_2(0,\cdot)\|_{L^2} \qquad \text{for all } t \in \mathbb{R}.$$

Pointwise control and proof that $q(t,z) \in L^4_{t,z}(\mathbb{R} \times \mathbb{C})$

$$\mathbf{s}(t,k) = e^{2i(k^2+\overline{k}^2)t}\mathbf{s}_0(k)$$

$$\begin{aligned} |q(t,z)| &= |\mathcal{S}^{-1}(\mathbf{s}(t,\cdot))(z)| \\ &\leq C(\|q_0\|_{L^2}) M\check{\mathbf{s}}(t,z) \end{aligned}$$

where

$$\check{\mathbf{s}}(t,z) = \int e_z(k) e^{2i(k^2+\overline{k}^2)t}\mathbf{s}_0(k)dk := U(t)(\check{\mathbf{s}}_0)(z)$$

is linear flow starting from $\check{\mathbf{s}}_0$ for which we have the Strichartz estimate

$$\|\check{\mathbf{s}}\|_{L^4_{t,z}} \lesssim \|\check{\mathbf{s}}_0\|_{L^2} = \|\mathbf{s}_0\|_{L^2} = \|q_0\|_{L^2}.$$

Time-domain Scattering

The Scattering Transform also yields the large time behaviour of the solutions to the DSII equation. Recall the definition of the wave operators, in the sense of nonlinear scattering theory.

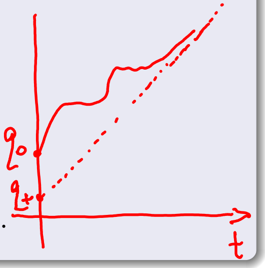
Definition

Let $q_0 \in L^2(\mathbb{R}^2)$ and let $q(t, z)$ be the solution to the Cauchy problem for defocusing DSII. Define $W_+ q_0 = q_+$ if there exists a unique $q_+ \in L^2(\mathbb{R}^2)$ such that

$$\lim_{t \to \infty} \|q(t, \cdot) - U(t) q_+\|_{L^2(\mathbb{R}^2)} = 0.$$

Similarly $W_- q_0 = q_-$ if

$$\lim_{t \to -\infty} \|q(t, \cdot) - U(t) q_-\|_{L^2(\mathbb{R}^2)} = 0.$$



Wave operators and asymptotic completeness for defocusing DSII

Theorem

a) The Wave operators $W_\pm$ for the defocusing DSII equation are well defined on every $q_0 \in L^2(\mathbb{R}^2)$ and

$$W_\pm q_0 = \check{\mathcal{S}} q_0.$$

b) The Wave operators $W_\pm$ are surjective, in fact norm-preserving diffeomorphisms of L^2.

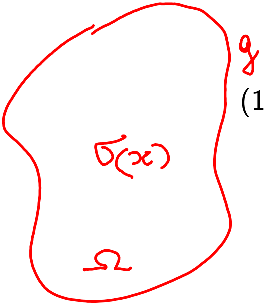
Perry (2014) established the same large time asymptotic behaviour in the L^∞ norm, for initial data in $H^{1,1} \cap L^1$.

An interesting consequence: the temporal scattering operator $W_+(W_-)^{-1}$ for defocusing DSII (i.e. the operator which sends q_- to q_+) is equal to the identity. The nonlinearity is invisible to observers at infinity.

The Calderón Inverse Conductivity Problem in Dimension 2

Let Ω be a simply connected domain in $\mathbb{R}^2 \simeq \mathbb{C}$

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega \\ u\Big|_{\partial\Omega} = g. \end{cases} \tag{12}$$



The Dirichlet-to-Neumann map is defined as

$$\Lambda_\sigma f := \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

A.P. Calderón (1980): does Λ_σ uniquely determine σ?

- N. (1996) - Unique reconstruction for $\sigma \in W^{2,p}(\Omega)$ for some $p > 1$
- R. Brown. G. Uhlman (1997) - $\sigma \in W^{1,p}(\Omega)$, for some $p > 2$.
- K. Knudsen, A.Tamasan (2005) - reconstruction for $\sigma \in C^{1+\varepsilon}(\Omega)$
- K. Astala, L. Päivärinta (2006) - $\sigma \in L^\infty$
- K. Astala, M. Lassas, L. Päivärinta (2016) - Larger class of conductivities which includes some unbounded ones.
- C.Carstea J.-N. Wang $\nabla \log \sigma \in L^2(\Omega)$ with small norm (2018)

From the Conductivity Equation to Pseudoanalytic Functions

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega \subset \mathbb{R}^2 \simeq \mathbb{C} \\ u\Big|_{\partial\Omega} = g. \end{cases} \tag{13}$$

We'll assume $\sigma > 0$ is such that $\nabla \log \sigma \in L^2(\Omega)$ and, for simplicity, $\sigma = 1$ on $\partial\Omega$.Then we prove that (13) is uniquely solvable for every $g \in H^1(\partial\Omega)$ and the Dirichlet-to-Neumann map is well-defined as a bounded operator $\Lambda_\sigma : H^1(\partial\Omega) \to L^2(\partial\Omega)$.

Let $v = \sigma^{\frac{1}{2}} \partial u$ then for u real valued, $\bar{\partial} v = q\bar{v}$ where $q = -\frac{1}{2}\partial \log \sigma \in L^2$, and on the boundary we have:

$$\frac{\partial u}{\partial \nu} = 2\Re(\nu v) \quad \text{and} \quad \frac{\partial u}{\partial \tau} = -2\Im(\nu v), \tag{14}$$

where $\nu = \nu_1 + i\nu_2$.

Hilbert Transform for Pseudoanalytic Functions

We are thus led to consider the following problem:

$$\begin{cases} \bar{\partial} v - q\bar{v} = 0 & \text{in } \Omega \\ \Im(\nu v) = g_0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

for $g_0 \in L^2(\partial\Omega)$ with integral zero, and to define an associated Hilbert Transform type operator on $\partial\Omega$ as:

$$\mathcal{H}_q(g_0) := \Re(\nu v). \tag{16}$$

(Astala and Päivärinta (2006) defined a similar Hilbert transform for the Beltrami equation). We have (see previous slide):

$$\Lambda_\sigma = -\mathcal{H}_q \frac{\partial}{\partial \tau}. \tag{17}$$

An Inverse Problem for Pseudoanalytic Functions

Theorem (N.-Regev-Tataru)

Assume $q \in L^2(\Omega)$ is as before. Then we can reconstruct q from knowledge of $\mathcal{H}_q$.

Connection to the Scattering Transform: (extend q to equal 0 outside Ω)

$$\begin{aligned}
\mathcal{S}q(k) &= \frac{1}{2\pi i}\int_{\mathbb{R}^2} e_k(z)\overline{q(z)}\Big(m_+(\cdot,k)+m_-(\cdot,k)\Big) \\
&= \frac{1}{2\pi i}\int_{\Omega} \partial\Big(\overline{m_+}(\cdot,k)-\overline{m_-}(\cdot,k)\Big) \\
&= \frac{1}{4\pi i}\int_{\partial\Omega} \overline{\nu}\Big(\overline{m_+}(\cdot,k)-\overline{m_-}(\cdot,k)\Big)
\end{aligned}$$

We prove that $\mathcal{H}_q$ determines the traces of $m_\pm(\cdot,k)$ on $\partial\Omega$.

An Exterior Problem

Define the functions:

$$\psi_\pm(z,k) = e^{izk} m_\pm(z,k). \tag{18}$$

The following lemma shows that we can obtain the trace $\psi_+(\cdot,k)\Big|_{\partial\Omega}$ from $\mathcal{H}_q$.

Lemma

Let Ω, q be as in the Inversion Theorem. Then the function $\psi_+(z,k)$ restricted to $z \in \mathbb{C}\backslash\overline{\Omega}$ is the unique solution of the exterior problem

$$\begin{cases} (i) & \bar\partial\psi_+ = 0 \ in \ \mathbb{C}\backslash\overline{\Omega} \\ (ii) & \psi_+(z,k)e^{-izk} - 1 \in L^4(\mathbb{C}\backslash\overline{\Omega}) \cap W^{1,\frac{4}{3}}_{loc} \\ (iii) & \Re(\nu\psi_+\Big|_{\partial\Omega}) = \mathcal{H}_q(\Im(\nu\psi_+\Big|_{\partial\Omega})). \end{cases} \tag{19}$$

The Calderón Inverse Conductivity Problem in Dimension 2

Theorem (N.-Regev-Tataru)

Suppose $\sigma > 0$ is such that $\nabla \log \sigma \in L^2(\Omega)$ and $\sigma = 1$ on $\partial\Omega$, then we can reconstruct σ from knowledge of Λ_σ.

Outline of the proof: with $q = -\frac{1}{2}\partial \log \sigma$ we have

$$\Lambda_\sigma \to \mathcal{H}_q \to m_\pm(\cdot, k) \text{ on } \partial\Omega \to \mathcal{S}q(k) \xrightarrow{\text{Plancherel}} q \to \sigma.$$

Thank you !