# Holonomy Inverse Problem

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# Summary

# In this talk

- $(M, g)$ is a compact Riemannian manifold without boundary; $\mathcal{E} \to M$ a vector bundle over $M$ equipped with a connection $\nabla^{\mathcal{E}}$. We address the following inverse problem:

## Question

*To what extent does the holonomy of $\nabla^{\mathcal{E}}$ over closed geodesics determine the gauge-equivalence class of $\nabla^{\mathcal{E}}$?*

## We will show

If $(M, g)$ has Anosov (chaotic) geodesic flow, $\mathcal{E}$ is Hermitian, and $\nabla^{\mathcal{E}}$ is unitary, only the traces of holonomy suffice to determine $[\nabla^{\mathcal{E}}]$ locally, and in some cases globally!

## Definition

A flow $\varphi_t : \mathcal{M} \to \mathcal{M}$ generated by a vector field $X$ is called Anosov if there is a continuous splitting $T\mathcal{M} = \mathbb{R}X \oplus E_u \oplus E_s$ into flow direction $\mathbb{R}X$, unstable/stable directions $E_{u/s}$ invariant under $d\varphi_t$, and there are constants $C, \nu > 0$ such that for all $x \in \mathcal{M}$, for some metric $|\cdot|$

$$|d\varphi_t(x)v| \leq \begin{cases} Ce^{-\nu t}|v|, & t \geq 0, v \in E_s(x), \\ Ce^{-\nu|t|}|v|, & t \leq 0, v \in E_u(x). \end{cases}$$

These flows model hyperbolic dynamics: sensitive (chaotic) upon a change in initial conditions. Restrictions on geometry/topology.

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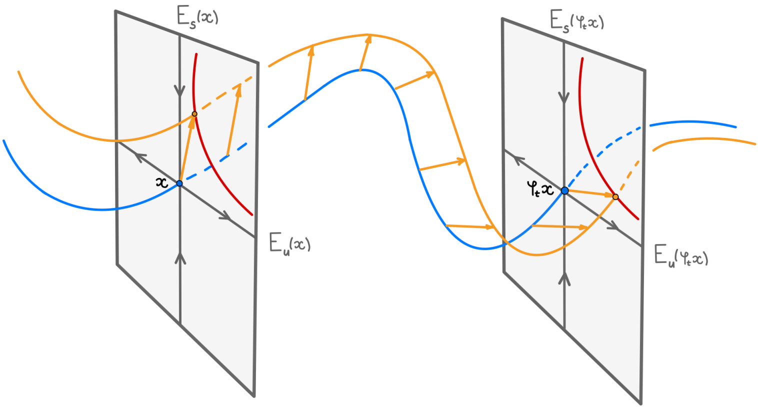
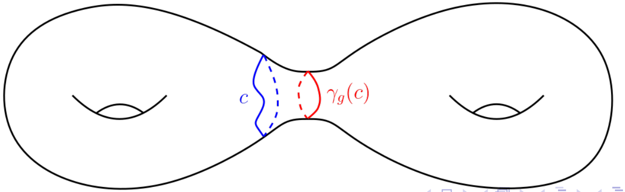


- Say that $(M, g)$ is Anosov if its geodesic flow is Anosov. Here

$$\mathcal{M} = SM = \{(x, v) \in TM : |v|_g = 1\}$$

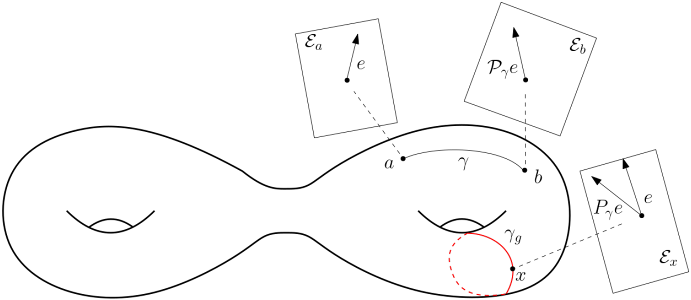
is the unit sphere bundle and $\varphi_t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$, where $\gamma_{x,v}(t)$ is the geodesic generated by the initial condition $(x, v)$.

- Examples:
  - If $(M, g)$ has negative sectional curvature, then it is Anosov.
  - $\exists$ examples with portions of positive curvature (Eberlein, Donnay-Pugh).
- If $(M, g)$ is Anosov, $\exists$ bijection between free homotopy classes $c \in \mathcal{C}$ and closed geodesics $\gamma_g(c)$ of length $L_g(c)$ in the class $c$.



# Recall: connections on vector bundles

- Connection $\nabla^{\mathcal{E}}$ is a map $\nabla^{\mathcal{E}} : C^\infty(M, \mathcal{E}) \to C^\infty(M, T^*M \otimes \mathcal{E})$ that locally looks like $d + A$ for a matrix $A$ of 1-forms.
- If $\gamma : [a, b] \to M$ a curve, $e \in \mathcal{E}_a$, $s : [a, b] \to \mathcal{E}$ is the parallel transport of $e$ along $\gamma$ if $\nabla^{\mathcal{E}}_{\dot{\gamma}} s = 0$ (first order ODE) and $s(a) = e$, $\pi \circ s = \gamma$. Denote $\mathcal{P}_\gamma e := s(b) \in \mathcal{E}_b$.



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- $\nabla^{\mathcal{E}}$ is unitary if compatible with the inner product on $\mathcal{E}$; it follows $\mathcal{P}_\gamma : \mathcal{E}_a \to \mathcal{E}_b$ is unitary.
- Denote the affine set of all connections on $\mathcal{E}$ by $\mathcal{A}_{\mathcal{E}}$. Gauge group $\mathcal{G}(\mathcal{E})$ is the set of all unitary isomorphisms of $\mathcal{E}$ and it acts on $\mathcal{A}_{\mathcal{E}}$ by pullback $p^*\nabla^{\mathcal{E}} := p^{-1}\nabla^{\mathcal{E}}(p\bullet)$. The quotient by $\mathcal{G}(\mathcal{E})$ is the moduli space, denoted by $\mathbb{A}_{\mathcal{E}} := \mathcal{A}_{\mathcal{E}}/\mathcal{G}(\mathcal{E})$. Two connections $\nabla^{\mathcal{E}}_1$ and $\nabla^{\mathcal{E}}_2$ are gauge-equivalent if there is a $p \in \mathcal{G}(\mathcal{E})$ such that $p^*\nabla^{\mathcal{E}}_2 = \nabla^{\mathcal{E}}_1$.
- Denote by $\mathbb{A} := \{([\mathcal{E}], [\nabla^{\mathcal{E}}])\}$ the moduli space of connections on all bundles over $M$.

# Primitive trace map

- Denote by $\mathcal{C}^\sharp = \{c_1^\sharp, c_2^\sharp, \dots\} \subset \mathcal{C}$ the set of *primitive* free homotopy classes, and by $\mathrm{Hol}_{\nabla^{\mathcal{E}}}(c^\sharp) \in \mathrm{U}(x_{c^\sharp})$ the parallel transport along $\gamma_g(c^\sharp)$ starting at some $x_{c^\sharp}$.
- $\mathrm{Hol}_{\nabla^{\mathcal{E}}}(c^\sharp)$ depends up to conjugation on the choice of both the point $x_{c^\sharp}$ and the equivalence class of the connection, but its trace does not.

## Definition

We define the primitive trace map as:

$$\mathcal{T}^\sharp : \mathbb{A} \ni ([\mathcal{E}], [\nabla^{\mathcal{E}}]) \mapsto \left(\mathrm{Tr}\left(\mathrm{Hol}_{\nabla^{\mathcal{E}}}(c_1^\sharp)\right), \mathrm{Tr}\left(\mathrm{Hol}_{\nabla^{\mathcal{E}}}(c_2^\sharp)\right), \dots\right) \in \ell^\infty(\mathcal{C}^\sharp).$$

## Question (Holonomy Inverse Problem)

*When is the primitive trace map $\mathcal{T}^\sharp$ injective?*

To study locally the problem, we will make the following assumptions:

**(A)** $\nabla^{\mathcal{E}}$ is **opaque**. By definition, this means that there are no non-trivial sub-bundles $\mathcal{F} \subset \mathcal{E}$ preserved by parallel transport along geodesics.

**(B)** *Generalized X-ray transform* $\Pi_1$ on twisted 1-forms with values in $\mathrm{End}(\mathcal{E})$ is **s-injective** (solenoidally injective).

### Theorem (C-Lefeuvre '21)

*Let $(M, g)$ be an Anosov manifold of dimension $\geq 3$ and $\mathcal{E} \to M$ a Hermitian vector bundle. Then, the primitive trace map $\mathcal{T}^\sharp$ is:*

(a) *locally injective near points in $\mathbb{A}$ satisfying (A) and (B),*

(b) *globally injective when restricted to direct sums of line bundles or to connections with small enough curvature.*

**Remark.** It was shown in our previous works that both conditions (A) and (B) are satisfied for an open and dense set of connections in the moduli space $\mathbb{A}$ in the $C^N$-topology.

# Remarks I

- Local injectivity: $\exists N \in \mathbb{N}$, such that $\mathcal{T}^\sharp$ is locally injective in the $C^N$-quotient topology on $\mathbb{A}_\mathcal{E}$; i.e. for any $[\nabla^\mathcal{E}] \in \mathbb{A}_\mathcal{E}$, there exists $\varepsilon > 0$ such that: for any $\nabla^\mathcal{E}_{1,2} \in \mathcal{A}_\mathcal{E}$ for which there are $p_{1,2} \in \mathcal{G}(\mathcal{E})$ with $\|p_i^* \nabla_i^\mathcal{E} - \nabla^\mathcal{E}\|_{C^N} < \varepsilon$, then $\mathcal{T}^\sharp(\nabla_1^\mathcal{E}) = \mathcal{T}^\sharp(\nabla_2^\mathcal{E})$ implies $[\nabla_1^\mathcal{E}] = [\nabla_2^\mathcal{E}]$.
- When $\dim M$ is odd, we also show that $\mathcal{T}^\sharp([\mathcal{E}], [\nabla^\mathcal{E}])$ determines $[\mathcal{E}]$.
- Example: if $M$ is a surface, then
  1. If $d$ is the trivial flat connection, $\mathcal{T}^\sharp([M \times \mathbb{C}], [d]) = (1, 1, \dots)$;
  2. If $\mathcal{K} = T^* M^{0,1}$ is the canonical line bundle equipped with the Chern connection $\nabla^{\mathrm{LC}}$, then $\mathcal{T}^\sharp([\mathcal{K}], [\nabla^{\mathrm{LC}}]) = (1, 1, \dots)$.
- **Paternain ['09, '10, '12, '13]** classified transparent connections on surfaces and showed their abundance on bundles with $\operatorname{rank} \mathcal{E} = 2$; see also **Guillarmou-Paternain-Salo-Uhlmann ['16]**.

# Remarks II

- Manifolds with boundary: studied with the convex foliation condition by **P-S-U-Zhou ['18]** and on simple surfaces **P-S-U ['12]**.
- Anosov embedding Theorem by **Chen-Erchenko-Gogolev ['20]** says that simple manifolds may be embedded into Anosov manifolds.
- Analogous marked length spectrum problem: study injectivity of $\mathcal{L}^\sharp : \mathbb{M}_{<0} \ni g \mapsto (L_g(c_1^\sharp), L_g(c_2^\sharp), \dots) \in \ell^\infty(\mathcal{C}^\sharp)$. Our approach similar in spirit to **Guillarmou-Lefeuvre ['19]**.

| | Marked Length Spectrum | Holonomy Inverse Problem |
|---|---|---|
| Object | metric $g$ | connection $\nabla^{\mathcal{E}}$ |
| Group Action | diffeomorphisms $\mathrm{Diff}_0(M)$ | gauge group $\mathcal{G}(\mathcal{E})$ |
| Data | $\mathcal{L}^\sharp : c \mapsto L_g(c)$ | $\mathcal{T}^\sharp : c \mapsto \mathrm{Tr}(\mathrm{Hol}_{\nabla^{\mathcal{E}}}(c))$ |
| Linearisation | $DL_g(c)(\beta) = \int_{\gamma_g(c)} \beta(\dot\gamma, \dot\gamma)$ | "$X$-ray on $\mathrm{End}(\mathcal{E})$-1-forms" |

- Length spectrum: the set of lengths of closed geodesics counted with multiplicities. We say the length spectrum is simple if all closed geodesics have distinct lengths (known to be a generic condition).
- Connection Laplacian is the operator $\Delta_{\mathcal{E}} := (\nabla^{\mathcal{E}})^* \nabla^{\mathcal{E}}$. It is 2nd order elliptic, self-adjoint, non-negative, acting on $C^\infty(M, \mathcal{E})$, with discrete spectrum $\mathrm{spec}(\Delta_{\mathcal{E}}) = \{0 \leq \lambda_0(\nabla^{\mathcal{E}}) \leq \lambda_1(\nabla^{\mathcal{E}}) \leq \dots\}$ counted with multiplicities.
- $\mathrm{spec}(\Delta_{\mathcal{E}})$ depends only on $[\nabla^{\mathcal{E}}]$ and hence we may define the spectrum map:

$$\mathcal{S} : \mathbb{A}_{\mathcal{E}} \ni [\nabla^{\mathcal{E}}] \mapsto \mathrm{spec}(\Delta_{\mathcal{E}}).$$

- Trace formula of Duistermaat-Guillemin applied to $\Delta_{\mathcal{E}}$ reads (assuming simple length spectrum, and $P_\gamma$ is the Poincaré map):

$$\lim_{t \to L_g(c)} (t - L_g(c)) \sum_{j \geq 0} e^{-it\sqrt{\lambda_j}} = \frac{L_g(c)\,\mathrm{Tr}\left(\mathrm{Hol}_{\nabla^{\mathcal{E}}}(c)\right)}{2\pi |\det(\mathrm{id} - P_{\gamma_g(c)})|^{1/2}}. \qquad (1.1)$$

- Consequence of (1.1) and the Main Theorem is:

**Corollary (C-Lefeuvre '21)**

*With the assumptions of the Main Theorem, the spectrum map $\mathcal{S}$ is:*

(a) *locally injective near any generic point $\mathfrak{a} \in \mathbb{A}$,*

(b) *globally injective when restricted to direct sums of line bundles or to connections with small enough curvature.*

- **Kuwabara ['90]**: counterexamples to injectivity of $\mathcal{S}$ for line bundles on covers of surfaces (simple length spectrum condition violated).
- Famous question of **Kac ['66]**: "*Can one hear the shape of a drum?*". Shape $\leftrightarrow$ magnetic field.
- Classical result of **Guillemin-Kazhdan ['80]**: $q \in C^\infty(M)$ determined from $\mathrm{spec}(-\Delta_g + q)$ (see also **Croke-Sharafutdinov ['98], P-S-U ['14]**).
- Our result is the first such for $\Delta_{\mathcal{E}}$ or more generally for an inverse spectral problem with an *infinite* gauge group.

# Summary

1 Introduction

2 Ideas of the Proof
- Dynamical result
- Parry's free monoid
- Moduli space of connections and Pollicott-Ruelle resonances

- Main new ingredients:
  - New Livšic-type theorem in hyperbolic dynamical systems with tight relation to representation theory, reducing the question to a transport problem on $\mathcal{M} = SM$;
  - Interplay between the geometry of the moduli space of connections and the theory of Pollicott-Ruelle resonances (microlocal analysis).
- Analogy: flat connections up to gauge correspond to representations of $\pi_1$ up to conjugacy; we will see that unitary connections up to dynamical (cocycle) equivalence correspond to representations of the Parry's free monoid.

- $\varphi_t : \mathcal{M} \to \mathcal{M}$ is a transitive Anosov flow and $\mathcal{E} \to \mathcal{M}$ a Hermitian vector bundle. Each $\nabla^{\mathcal{E}} \in \mathcal{A}_{\mathcal{E}}$ gives rise to a unitary cocycle $C(x, t) : \mathcal{E}_x \to \mathcal{E}_{\varphi_t x}$ by parallel transport (i.e. it satisfies $C(\varphi_t x, t')C(x, t) = C(x, t + t')$).
- Our new Livšic-type result in hyperbolic dynamical systems:

**Theorem (C-Lefeuvre '21)**

*Let $\mathcal{E}_{1,2} \to \mathcal{M}$ be vector bundles equipped with unitary connections $\nabla^{\mathcal{E}}_{1,2}$, which induce unitary cocycles $C_{1,2}$ via parallel transport. Assume that for each primitive closed orbit $\gamma \ni x$ of period $T$ we have*

$$\mathrm{Tr}(C_1(x, T)) = \mathrm{Tr}(C_2(x, T)).$$

*Then $\exists\ p \in \mathcal{G}(\mathcal{E}_2, \mathcal{E}_1)$ such that for all $x \in \mathcal{M}, t \in \mathbb{R}$:*

$$C_1(x, t) = p(\varphi_t x)C_2(x, t)p(x)^{-1}.$$

# Some remarks

- In particular we have $\mathcal{E}_1 \cong \mathcal{E}_2$ via the map $p$.
- Result goes back to **Livšic ['72]**: if $f \in C^\infty(\mathcal{M})$ integrates to zero along every closed geodesic, then $\exists u \in C^\alpha(\mathcal{M})$ such that $Xu = f$ (Abelian cocycle). Inspired by **Parry ['99]** and **Schmidt ['99]** who show a weaker result.
- When $\mathcal{M} = SM$, $\mathcal{E}_{1,2}$ are pullbacks of bundles from $M$, then by the Theorem and differentiating in time, we get

$$\nabla_X^{\mathrm{Hom}(\mathcal{E}_2, \mathcal{E}_1)} p = 0.$$

*Assuming* $p$ depends only on $x$-variable, we see that $p^* \nabla^{\mathcal{E}_1} = \nabla^{\mathcal{E}_2}$.

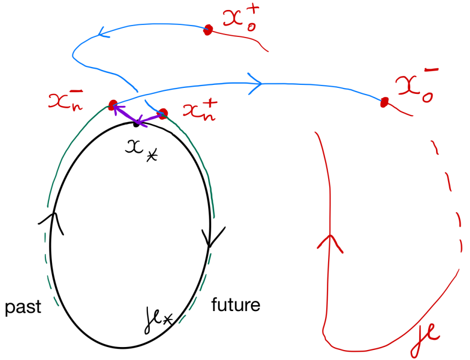
- Let $x_\star \in \mathcal{M}$ be a periodic point. We say $p$ is homoclinic to $x_\star$ if $d(\varphi_{t \pm t_0^\pm} p, \varphi_t x_\star) \to_{t \to \pm\infty} 0$ for some $t_0^\pm \in \mathbb{R}$; similarly an orbit $\gamma$ is homoclinic to $x_\star$ if it contains a point homoclinic to $x_\star$.
- Let $\mathcal{H}$ be the set of all homoclinic orbits. Using the shadowing property for Anosov flows, homoclinic orbits are dense.
- We introduce the Parry's free monoid as the monoid generated by $\mathcal{H}$, i.e. the formal set of words (empty word corresponds to $\mathbf{1}_{\mathbf{G}}$):

$$\mathbf{G} := \{\gamma_1^{m_1} \ldots \gamma_k^{m_k} \mid k \in \mathbb{N}, m_1, \ldots, m_k \in \mathbb{N}_0, \gamma_1, \ldots, \gamma_k \in \mathcal{H}\},$$

Fix $\gamma \in \mathcal{H}$ and set $x_n^{\pm} := \gamma(A_{\pm} \pm nT_{\star})$. Take a sequence $k_n \to \infty$ such that $C(x_\star, T_\star)^{k_n} \to \mathrm{id}_{\mathcal{E}_{x_\star}}$ and set

$$\rho_n(\gamma) := C_{x_{k_n}^+ \to x_\star} C(x_0^+, k_n T_\star) C(x_0^-, T_\gamma) C(x_{k_n}^-, k_n T_\star) C_{x_\star \to x_{k_n}^-}.$$

Define $\rho(\gamma) := \lim_{n\to\infty} \rho_n(\gamma)$; get a representation $\rho : \mathbf{G} \to \mathrm{U}(\mathcal{E}_{x_\star})$.



# Sketch-proof of the dynamical Theorem

**Lemma**

*If $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$ are trace-equivalent, then the induced representations $\rho_{1,2} : \mathbf{G} \to \mathrm{U}(\mathcal{E}_{x_\star})$ are isomorphic, i.e. there is a $p_\star \in \mathrm{U}(\mathcal{E}_{2x_\star}, \mathcal{E}_{1x_\star})$:*

$$\forall \gamma \in \mathbf{G}, \quad \rho_1(\gamma) = p_\star \rho_2(\gamma) p_\star^{-1}.$$

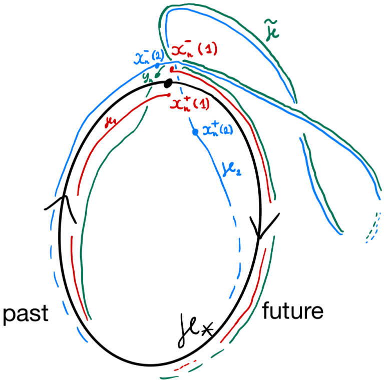
- By algebra, it suffices to show $\rho_1, \rho_2$ have equal characters. Take $\gamma_{1,2} \in \mathcal{H}$, $\gamma = \gamma_1 \cdot \gamma_2$ and show $\mathrm{Tr}(\rho_1(\gamma)) = \mathrm{Tr}(\rho_2(\gamma))$. We have

$$\rho_1(\gamma) = \rho_{1,n}(\gamma_1)\rho_{1,n}(\gamma_2) + o(1).$$

By the shadowing property, take $\widetilde{\gamma} \ni y_n$ that $\mathcal{O}(e^{-\theta k_n})$-shadows the concatenation $S = [x_{k_n^-}(1) x_{k_n^+}(1)] \cup [x_{k_n}^-(2) x_{k_n}^+(2)]$. Thus:

$$\rho_{1,n}(\gamma_1)\rho_{1,n}(\gamma_2) = C_{1,y_n \to x_\star} C_1(y_n, T_n') C_{1,y_n \to x_\star}^{-1} + \mathcal{O}(e^{-\theta k_n}).$$

- Taking traces and letting $n \to \infty$, we get the claim.




The proof is completed by pushing $p_\star$ along elements of $\mathcal{H}$ by parallel transport with respect to the connection $\nabla^{\mathrm{Hom}(\nabla^{\mathcal{E}_2},\nabla^{\mathcal{E}_1})}$ from both the past and the future. Both pushforwards agree by construction; denote them by $p$. We show $p$ is Lipschitz continuous, so by a regularity result of **Bonthonneau-L ['21]** $\implies$ $p$ is smooth.

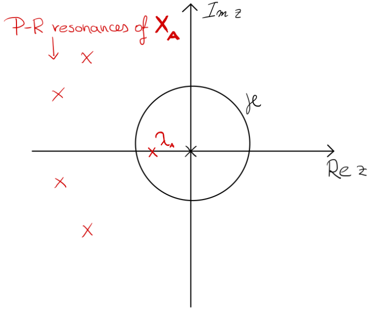
- Let $[\nabla^{\mathcal{E}}] \in \mathbb{A}_{\mathcal{E}}$; consider $A \in C^\infty(M, \mathrm{End}_{\mathrm{sk}}(\mathcal{E}))$. Consider the operator $\mathbf{X}_A := (\pi^* \nabla^{\mathrm{Hom}(\nabla^{\mathcal{E}}+A, \nabla^{\mathcal{E}})})_X$. By opacity of $\nabla^{\mathcal{E}}$, we know that $\mathbf{X} = \mathbf{X}_0$ has a simple resonance at zero (spanned by $\mathrm{id}_{\mathcal{E}}$).
- P-R resonances are poles of the meromorphic extension of $(\mathbf{X} + z)^{-1} : C^\infty \to \mathcal{D}'$. By continuity, $\exists$ small contour $\gamma$ such that for $A$ small enough, no resonance crosses $\gamma$. Set:

$$\Pi_A^+ := \frac{1}{2\pi i} \int_\gamma (z + \mathbf{X}_A)^{-1} dz, \quad \lambda_A := \mathrm{Tr}(-\mathbf{X}_A \Pi_A^+).$$

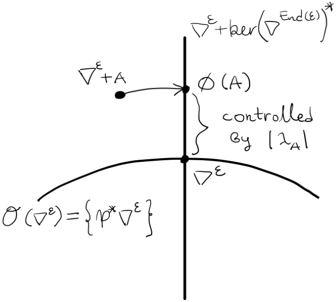
- Denote by $\phi(A)$ the gauge-equivalent connection sending $\nabla^{\mathcal{E}} + A$ to Coulomb gauge, $(\nabla^{\mathrm{End}(\mathcal{E})})^*(\phi(A) - \nabla^{\mathcal{E}}) = 0$.
- It turns our that the second variation controls the distance in the moduli space (convexity):

$$0 \leq \|\phi(A) - \nabla^{\mathcal{E}}\|^2_{H^{-1/2}(M, T^*M \otimes \mathrm{End}_{\mathrm{sk}}(\mathcal{E}))} \leq C|\lambda_A|.$$

## Complex plane




## Moduli space



Thank you for your attention!