Determining a Riemannian metric from least-area data

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Outline

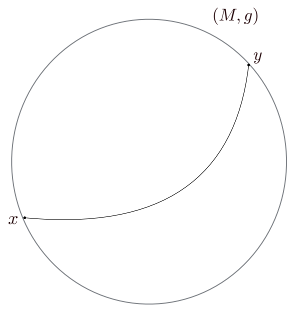
1. Historical context
2. A lower codimensional rigidity problem
3. Sketch of the proof of the global result
4. Future projects

Historical context

A classical geometric question

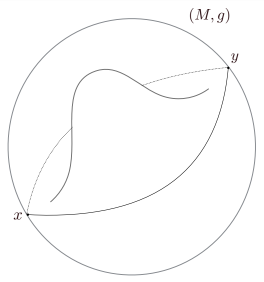
The boundary rigidity problem

- (M, g) a Riemannian manifold with boundary ∂M.
- Know the geodesic distance between any two boundary points $x, y \in \partial M$.
- Does this information determine the Riemannian metric g?



Obstructions

- Boundary-fixing diffeomorphisms.
- Regions of large positive curvature.



- Manifolds without such regions are called **simple**.
- Conjecture (Michel 1981): All simple manifolds are boundary rigid.

Selected results on boundary rigidity

- Special cases were shown by Michel, Gromov, and Croke.
- Lassas, Sharafutdinov, Uhlmann (2003): g is C^k-close to Euclidean.
- Stefanov and Uhlmann (2005): g, $\tilde{g}$ are simple and $\tilde{g}$ is C^k-close to g.
- Pestov and Uhlmann (2005): **Simple 2-manifolds are boundary rigid.**
- Burago and Ivanov (2010 and 2013): g is simple and either C^2-close to Euclidean or C^3-close to a hyperbolic metric.
- Graham, Guillarmou, Stefanov, Uhlmann (2019): Asymptotically hyperbolic setting.
- Stefanov, Uhlmann, and Vasy (2021): Manifolds with a convex foliation condition + lens data.

2D rigidity

Pestov-Uhlmann (2005):

- (M, g) simple 2D manifold.
- knowledge of boundary distances equivalent to knowledge of Dirichlet-to-Neumann map for a conductivity-type problem.
- Lassas-Uhlmann (2001):

$$\begin{aligned} \Delta_g u &= 0 \text{ on } M \\ u &= f \text{ on } \partial M. \end{aligned}$$

  The **Dirichlet-to-Neumann** map $\Lambda_g : f \to g(\nabla u, \nu)|_{\partial M}$ uniquely determines g. Here ν outward-pointing unit normal to ∂M.
- This settled the 2D boundary rigidity problem.

2D Calderón's Problem

Nachman (1996):

For anisotropic case,

- Write $\gamma_{ij} = \sqrt{\det g}\, g_{ij}^{-1}$.
- Then $\nabla \cdot (\gamma(x)\nabla u) = 0$ transforms to $\Delta_g u = 0$.
- Isothermal coordinates $g = e^{2\phi(x)} I_{2\times 2}$ reduces to isotropic case.

For isotropic case,

- Write $q = \frac{\Delta\gamma}{\sqrt{\gamma}}$, $\gamma \in C^2(M)$.
- $\nabla \cdot (\gamma(x)\nabla u) = 0$ transforms to $\Delta u + qu = 0$.
- Reconstructed q from DN-map $\Lambda : f \to \nu \cdot \nabla u$.

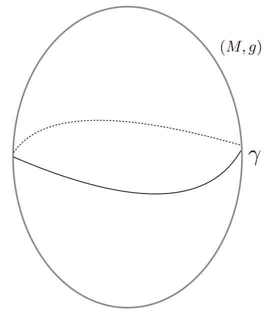
A lower codimensional rigidity problem

Determining the metric from area data

- Let us consider a codimension $n - 2$ version of boundary rigidity.
- Consider least-areas of minimal surfaces instead of distances of geodesics.

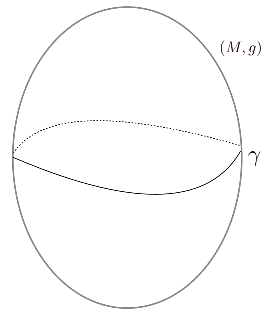
Question

- (M, g) a Riemannian manifold with boundary ∂M.
- For any simple closed curve $\gamma \subset \partial M$, we know the area of the least-area surface(s) circumscribed by γ.
- Does this information determine the Riemannian metric?



Answer

- **Yes!** (under certain geometric conditions.)
- In some cases, we only require the area data for a much smaller subclass of curves.



Interesting connection to physics

AdS/CFT theories

- AdS/CFT correspondence:
  Relates quantum gravity defined on an asymptotically Anti-de Sitter (AdS) spacetime to a conformal field theory (CFT) defined on the conformal boundary.

- Hubeny-Ryu-Takayanagi Conjecture:
  Entanglement entropy of a region A in the CFT
  $\longleftrightarrow$ area of a least-area surface $Y \subset$ AdS with boundary $\partial Y = \partial A$.

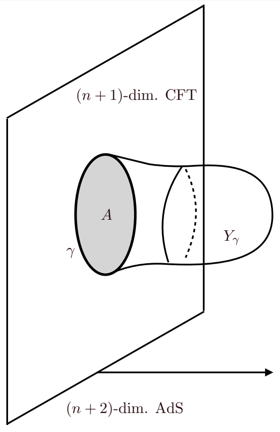


Figure 1: Region A in an $(n+1)$-dimensional CFT and a least-area surface Y_γ in $(n+2)$-dimensional AdS.

AdS/CFT theories

In AdS/CFT, expect knowledge of the boundary determines the bulk.

1. Is this true?
2. What other features of the bulk could you identify?

Physics work which builds on our results

- N. Bao, CJ. Cao, S. Fischetti, C. Keeler. *Towards bulk metric reconstruction from extremal area variations*, 2019 Class. Quantum Grav. 36 185002.
- N. Bao, CJ. Cao, S. Fischetti, J. Pollack, Y. Zhong. *More of the bulk from extremal area variations*, 2021 Class. Quantum Grav. 38 047001.

Our main results on recovering a Riemannian metric from area data:

We can determine a Riemannian metric from knowledge of least-areas for **three** classes of manifolds.

Briefly:

- The first two classes of manifolds arise from the tradeoff:
  less area data available $\longrightarrow$ more restrictions on the geometry.
- The third class of manifolds arise from the tradeoff:
  more data available $\longrightarrow$ fewer restrictions on the metric.

I will discuss a result for classes 1 and 2 today.

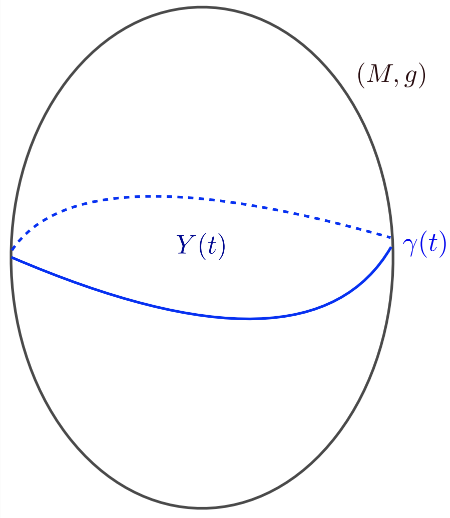
Our global result

Theorem (Alexakis, B., Nachman, 2020)

- (M, g) a manifold of Class 1 or Class 2.
- $g|_{\partial M}$ given.
- Suppose for the given family of simple closed curves $\gamma(t) \subset \partial M$ and any nearby perturbations $\gamma(s,t) \subset \partial M$, we know the area of the properly embedded surface $Y(s,t) \subset M$ which solves the least-area problem for $\gamma(s,t)$.

Then, the metric g is uniquely determined up to diffeomorphisms which fix ∂M.

Set up



Determining a Riemannian metric from least-area surfaces:

The **first** and **second** class of manifolds:

Let (M, g) be a Riemannian manifold with boundary ∂M satisfying

- (M, g) is C^4-smooth.
- $\dim(M) = 3$.
- (M, g) has strictly mean convex boundary ∂M.
- there is a foliation of ∂M by simple closed curves $\{\gamma(t)\}_{t\in(-1,1)}$ which satisfy some technical curvature bounds.
- the foliation $\{\gamma(t)\}_{t\in(-1,1)}$ induces a foliation of M by area minimizing discs $\{Y(t)\}_{t\in(-1,1)}$.

Class 1: For (M, g) as described, we additionally have g is C^3-close to Euclidean.

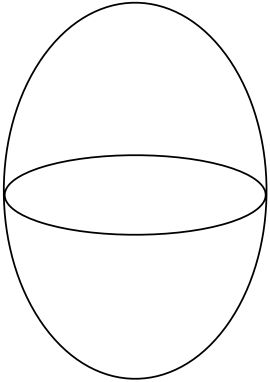


Figure 2: g “looks flat” even when zoomed to level of curvature.

Class 2:

Class 2: For (M, g) as described, (M, g) is also **straight-thin**: the minimal surfaces $Y(t)$ have area bounded above by a (small) number and (M, g) is not too "curvy".

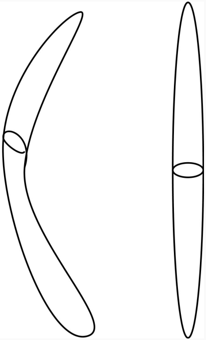


Figure 3: Cross-sectional area is small.

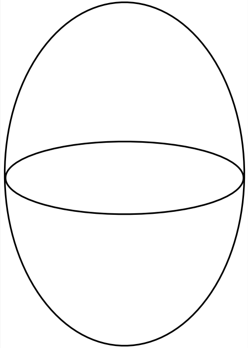
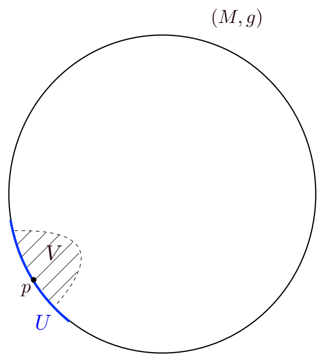


Figure 4: Wider cross-section compensated by "straightness".

Local result

- What if we only know area info near a point on the boundary?
- Could we determine the metric near the point?



Local result

Theorem (Alexakis, B., Nachman)

- (M, g) a 3-dimensional Riemannian manifold with boundary ∂M.
- ∂M is both C^4-smooth and mean convex at $p \in \partial M$.
- $U \subset \partial M$ is a neighbourhood of p with $g|_U$ known, and a given foliation $\{\gamma(t)\}_{t\in(-1,1)}$ of U by simple, closed curves.
- Suppose that for $\gamma(t)$ and any nearby perturbation $\gamma(s,t) \subset U$, we know the area of the properly embedded surface $Y(s,t)$ which solves the least-area problem for $\gamma(s,t)$.

Then, there exists a neighbourhood $V \subset M$ of p such that g is uniquely determined on V up to isometries which fix $V \cap \partial M$.

Sketch of the proof of the global result

Overview of global result proof:

Want to show: least-area data for the foliation $\{Y(t) : t \in (-1,1)\} = M$ and its nearby perturbations $\implies g$ is uniquely determined.

- Solve for the metric by moving along the foliation $Y(t)$.
- Use conformal structure of each $Y(t)$ to write the metric as

$$g = \begin{pmatrix} e^{2\phi} & 0 & g_{31} \\ 0 & e^{2\phi} & g_{32} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}.$$

- Note: by extending (M, g) to an asymptotically flat manifold, ϕ is unique on each $Y(t)$.

Main proof ideas:

- **Key:** Use variations of the foliation to relate geometric data to PDE data.
- By considering the normal variation of $Y(0)$ to $Y(t)$, we find

$$\frac{\partial^2}{\partial t^2} A(Y(t))\Big|_{t=0} = -\int_{Y(0)} \psi \left(\Delta_{Y(0)} + \mathrm{Ric}^M(\vec{n},\vec{n}) + ||A||^2 \right) \psi \, d\mathrm{Vol} + \int_{\partial Y(0)} \psi g(\nabla \psi, \nu)\, dS + \int_{\partial Y(0)} g(\nabla_V V, \nu)\, dS.$$

where $V = \psi \vec{n}$, $\psi : Y(0) \to \mathbb{R}$, and $\vec{n}$ is a unit normal vector field.

Main proof ideas:

- In our conformal coordinates, we determine the **Dirichlet-to-Neumann map**

$$\Lambda_{g_{\mathbb{E}}} : \psi_0 \mapsto \frac{\partial \psi}{\partial \nu}$$

for

$$\begin{aligned} \Delta_{g_{\mathbb{E}}} \psi + e^{2\phi} \left(\mathrm{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2 \right) \psi &= 0 \quad \text{on } D \subset \mathbb{R}^2 \quad (1) \\ \psi &= \psi_0 \quad \text{on } \partial D. \end{aligned}$$

- **Nachman (1996):**

  $\Lambda_{g_{\mathbb{E}}}$ determines $e^{2\phi} \left(\mathrm{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2 \right)$.
- Thus we know **any** solution ψ to (1).

Main proof ideas:

- For the foliation $\{Y(t)\}_{t\in(-1,1)}$, the lapse function $\psi := ||N||_g$ is a solution to (1).

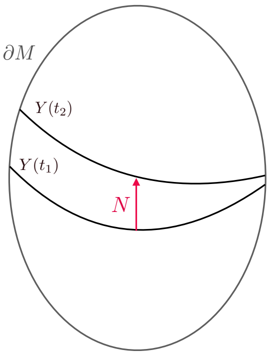
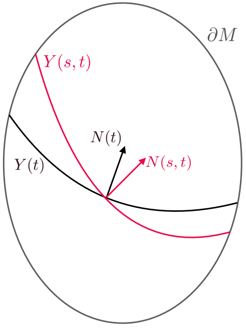


Figure 5: The lapse function is $||N||_g$.

Main proof ideas:

- Variations $Y(s,t)$ of $Y(t)$ lead to knowledge of **new** lapse functions $\psi(s,t) := ||N(s,t)||_g$.



Main proof ideas:

Next steps:

- Linearizing $||N(s,t)||_g$ about $s = 0$ gives nonlinear, non-local equations for the components of g^{-1}.
- Get an evolution equation for ϕ from the minimality of each $Y(t)$.
- We show uniqueness for this system by considering two metrics g_1 and g_2 for which we have the same area data.

Main proof ideas:

- Obtain $\delta g^{33} := g_1^{33} - g_2^{33} = 0$ in the coordinates (x^α).
- Taking differences of the equations we derived:

$$
\begin{aligned}
0 &= \delta g^{31}(p) + \partial_k ||\nabla x^3||_{g_1}(p) \delta \dot{x}_1^k(p) \\
0 &= \delta g^{32}(p) + \partial_k ||\nabla x^3||_{g_1}(p) \delta \dot{x}_2^k(p) \\
0 &= g_1^{k3} \partial_k(\delta\phi) + g_1^{33} \partial_3(\delta\phi) \\
&\qquad + \left(\partial_k \phi_2 - \frac{1}{2} \partial_k \log(g_1^{33}) \right) \delta g^{k3} + \frac{1}{2} \partial_k (\delta g^{3k}).
\end{aligned}
$$

  in the differences δg^{31}, δg^{32}, and $\delta\phi$.
- Here $\delta \dot{x}_i^k$ is a pseudodifferential operator (ΨDO) acting on δg^{31}, δg^{32}, $\delta\phi$ and $\partial_3 \delta\phi$.

Main ideas of the proof

- We show δg^{31}, δg^{32} are ΨDOs acting on $\delta\phi$ and $\partial_3\delta\phi$.
  - The conditions of close to Euclidean or straight-thin are used to invert the system.
- Then, the equation for $\delta\phi$ becomes a hyperbolic Cauchy problem:

$$\begin{aligned} \partial_3\delta\phi + P(\delta\phi) = 0 \quad & \text{on } M \\ \delta\phi = 0 \quad & \text{on } \partial M. \end{aligned}$$

  where P is an order 1 ΨDO in the tangential directions.
- The uniqueness of this Cauchy problem gives us uniqueness of the metric components.

Class 3: (M, g) admits foliations from all directions

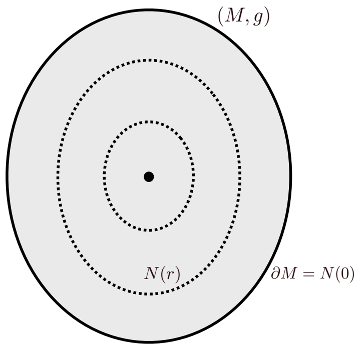


Figure 6: Foliations by mean convex submanifolds $N(r)$.

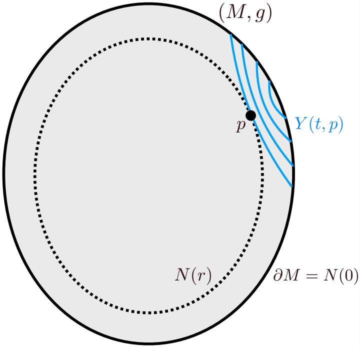


Figure 7: Foliations by area-minimizers which reach $p \in M$.

Global result II

Theorem (Alexakis, B., Nachman)

- Suppose (M, g) admits foliations from all directions.
- $g|_{\partial M}$ given.
- Suppose that for all $p \in M$ and for each $\gamma(t, p)$ as above, and any nearby perturbation $\gamma(s, t, p) \subset \partial M$, we know the area of the properly embedded surface $Y(s, t, p)$ which solves the least-area problem for $\gamma(s, t, p)$.

Then the knowledge of these areas uniquely determines the metric g (up to isometries which fix the boundary).

Future projects

Future projects

- AdS/CFT - renormalized area information.
- Larger classes of 3-manifolds.

Thanks!

$\dot{x}^k$ equation

$$
\begin{aligned}
\Delta_{g_0}\dot{x}^k &= -2\psi_p A^{ij}\Gamma^k_{ij}(g_0) - 2g_0^{ij}\nabla_j(\psi_p A^k_i) \\
&= -g_0^{ij}\nabla_j(\psi_p)\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ij}) \\
&\quad + 2g_0^{ij}\psi_p\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ij}) \\
&\quad + g_0^{ij}\psi_p\frac{1}{||\nabla x^3||_g}\partial_j||\nabla x^3||_g\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{2||\nabla x^3||_g}(g_{\alpha j}\partial_i g^{3\alpha} + g_{i\alpha}\partial_j g^{3\alpha} + g^{3\alpha}\partial_\alpha g_{ij}) \\
&\quad - g_0^{ij}\psi_p\frac{e^{-2\phi}(g_{\mathbb{E}})^{jk}}{||\nabla x^3||_g}\Big\{\partial_j g_{\alpha m}\partial_i g^{3\alpha} + g_{\alpha m}\partial_j\partial_i g^{3\alpha} + \partial_j g_{\alpha i}\partial_m g^{3\alpha} + g_{\alpha i}\partial_j\partial_m g^{3\alpha} \\
&\quad + 2e^{2\phi}(g_{\mathbb{E}})_{im}\partial_j g^{3\alpha}\partial_\alpha\phi + 2e^{2\phi}(g_{\mathbb{E}})_{im}g^{3\alpha}\partial_j\partial_\alpha\phi - 4e^{2\phi}(g_{\mathbb{E}})_{im}g^{3\alpha}\partial_\alpha\phi\partial_j\phi\Big\} \\
&\quad - 8\psi_p e^{-4\phi}\Big\{g_{\mathbb{E}}^{km}g_{\mathbb{E}}^{jl}g_{3l}\partial_m g^{33}\partial_j\phi + g_{\mathbb{E}}^{im}g_{\mathbb{E}}^{kl}g_{3l}\partial_m g^{33}\partial_i\phi - g_{\mathbb{E}}^{kj}g_{\mathbb{E}}^{ml}g_{3l}\partial_m g^{33}e^{2\phi}\partial_j\phi \\
&\quad + g_{\mathbb{E}}^{km}e^{2\phi}\partial_m g^{3j}\partial_j\phi + g_{\mathbb{E}}^{im}e^{2\phi}\partial_m g^{3k}\partial_i\phi - g_{\mathbb{E}}^{kj}\partial_m g^{3m}e^{4\phi}\partial_j\phi\Big\} \\
&=: \mathcal{F}^k(g^{13}, g^{23}, \phi, \psi_{p,i}, d\psi_{p,i}, p).
\end{aligned}
$$

$\delta\dot{x}^k$ equation

Here $\delta\dot{x}_i^k$ is a pseudodifferential operator (ΨDO) acting on δg^{31}, δg^{32}, $\delta\phi$ and $\partial_3\delta\phi$:

$$
\begin{aligned}
\Delta_{g_{\mathbb{E}}}\delta\dot{x}^k = {} & \psi_{p,i}\bar{A}_m^{jkl}\partial_l\partial_j\delta g^{3m}(w) + \psi_{p,i}\bar{B}^{jk\alpha}\partial_j\partial_\alpha\delta\phi(w) \\
& + (\psi_{p,i}\bar{C}_1^{k\alpha} + \partial_j\psi_{p,i}\bar{C}_2^{jk\alpha})(w)\partial_\alpha\delta\phi(w) \\
& + (\psi_{p,i}\bar{C}_3 + \partial_j\psi_{p,i}\bar{C}_4^j)\delta\phi \\
& + (\psi_{p,i}\bar{D}_{1m}^{jk} + \partial_l\psi_{p,i}\bar{D}_{2m}^{jkl})(w)\partial_j\delta g^{3m}(w) \\
& + (\psi_{p,i}\bar{F}_{1m}^{k} + \partial_l\psi_{p,i}\bar{F}_{2m}^{kl})(w)\delta g^{3m}(w),
\end{aligned}
$$

for smooth functions $\bar{A}_m^{jkl}, \ldots, \bar{F}_{2m}^{kl}$ in the unknown metric coefficients g_1^{13}, g_1^{23} and g_2^{13}, g_2^{23} and their first and second derivatives at q.