Stability of an inverse problem for a semi-linear wave equation

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Setting

Let (N, g) be an $n+1$-dimensional $(n \geq 2)$ globally hyperbolic Lorentzian manifold. We assume that $N = \mathbb{R} \times M$ and the metric tensor is of the form

$$g = -\beta(t,x)dt^2 + h(t,x),$$

where $\beta > 0$ is a smooth function and $h(t, \cdot)$, $t \in \mathbb{R}$, is a smooth one-parameter family of Riemannian metrics on an n-dimensional manifold M.

Let $\Box_g$ be the D'Alembertian wave operator

$$\Box_g u = -\sum_{a,b=0}^{n} \frac{1}{\sqrt{|\det(g)|}} \frac{\partial}{\partial x^a} \left(\sqrt{|\det(g)|} g^{ab} \frac{\partial u}{\partial x^b} \right).$$

- Example: $\Box_{\mathrm{Minkowski}} = \partial_t^2 - \Delta_x$ in $\mathbb{R} \times \mathbb{R}^n$.

Notation and setting

Let $\Omega \subset M$ be a smooth compact submanifold with boundary (possibly non-convex).
We are interested in the recovery of the potential function q in the nonlinear wave equation

$$\begin{cases} \Box_g u(t,x) + q(t,x)u(t,x)^m = 0, & \text{in } [0,T] \times \Omega, \\ u = f, & \text{on } [0,T] \times \partial\Omega, \\ u(0,x) = \partial_t u(0,x) = 0, & \text{on } \Omega, \end{cases}$$

We assume that the exponent m is an integer and $m \geq 4$.

Notation and setting

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We will let $\Sigma = [0,T] \times \partial\Omega$ denote the lateral boundary of the domain.
The Dirichlet-to-Neumann map (DN map) is defined by

$$\Lambda : H_c^{s+1}(\Sigma) \to H^s(\Sigma),$$
$$\Lambda(f) = \partial_\nu u_f\big|_\Sigma,$$

where ν is the outward normal vector to Σ.

The inverse problem

$$\begin{cases} \Box_g u(t,x) + q(t,x)u(t,x)^m = 0, & \text{in } [0,T]\times\Omega, \\ u = f, & \text{on } [0,T]\times\partial\Omega, \\ u(0,x) = \partial_t u(0,x) = 0, & \text{on } \Omega, \end{cases}$$

Inverse Problems

Recover $q(x,t)$ from the DN map Λ. How stable is this reconstruction process?

Some earlier results

- Uniqueness for an inverse problem for $(\partial_t^2 - c(x)^2\Delta)u = 0$ by boundary control methods by Belishev, Belishev-Kurylev in 1987-1995
- First results exploiting nonlinearity of the equation by Kurylev, Lassas and Uhlmann from 2014
- Since then, techniques using nonlinearity as a tool have been extremely popular: T Balehowsky, C Cârstea, X Chen, M de Hoop, A Feizmohammadi, C Guillarmou, P Hintz, Y Kian, H Koch, K Krupchyk, M Lassas, T Liimatainen, Y-H Lin, G Nakamura, L Oksanen, G Paternain, A Rüland, M Salo, P Stefanov, G Uhlmann, Y Wang, J Zhai, and many more (apologies to any who I missed!)

The inverse problem

$$\begin{cases} \Box_g u(t,x) + q(t,x) u(t,x)^m = 0, & \text{in } [0,T]\times\Omega, \\ u = f, & \text{on } [0,T]\times\partial\Omega, \\ u(0,x) = \partial_t u(0,x) = 0, & \text{on } \Omega, \end{cases}$$

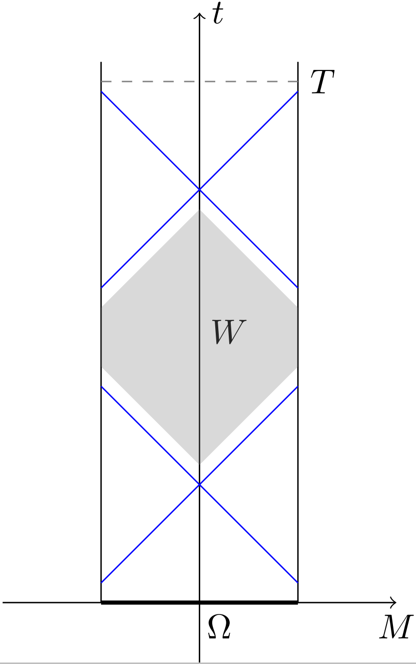
Inverse Problems

Recover $q(x,t)$ from the DN map Λ. How stable is this reconstruction process?

The finite speed of propagation of waves causes natural limitation to what we can recover: Let

$$W \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0,T]\times\Omega)$$

be a compact set.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Theorem (Lassas, Liimatainen, Potenciano-Machado,T 2021)

Assume that $q_1, q_2 \in C^{s+1}(\mathbb{R} \times \Omega)$ *satisfy* $\|q_j\|_{C^{s+1}} \leq c$, $j = 1, 2$, *for some* $c > 0$ *and* $s > (n-1)/2$. *Let* $\Lambda_1, \Lambda_2 : H_c^{s+1}(\Sigma) \to H^r(\Sigma)$, $r \leq s$, *be the corresponding Dirichlet-to-Neumann maps of the non-linear wave equation.*

Let $\varepsilon_0 > 0$, $L > 0$ *and* $\delta \in (0, L)$ *be such that*

$$\|\Lambda_1(f) - \Lambda_2(f)\|_{H^r(\Sigma)} \leq \delta$$

for all $f \in H_0^{s+1}(\Sigma)$ *with* $\|f\|_{H^{s+1}(\Sigma)} \leq \varepsilon_0$. *Then there exists a constant* $C > 0$, *independent of* q_1, q_2 *and* $\delta > 0$, *such that*

$$\|q_1 - q_2\|_{L^\infty(W)} \leq C\delta^{\sigma(s,m)},$$

where

$$\sigma(s, m) = \frac{8(m-1)}{2m(m-1)(8s - n + 13) + 2m - 1}.$$

Ideas of the proof: higher-order linearization

Use small parameters $\varepsilon_1, \dots, \varepsilon_m > 0$ and differentiate:

$$\begin{cases} \Box_g u(t,x) + q(t,x)u(t,x)^m = 0, & \text{in } [0,T] \times \Omega, \\ u = \varepsilon_1 f_1 + \cdots + \varepsilon_m f_m, & \text{on } [0,T] \times \partial\Omega, \\ u(0,x) = \partial_t u(0,x) = 0, & \text{on } \Omega, \end{cases}$$

Noisy measurements blow up, when using actual derivatives (Frechet derivatives)

$$\frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_m}\Big|_{\varepsilon_1 = \ldots = \varepsilon_m = 0}.$$

Instead, we will use *finite differences*

$$D^m_{\vec{\varepsilon}} u_{\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m} := \frac{1}{\varepsilon_1 \cdots \varepsilon_m} \sum_{\sigma \in \{0,1\}^m} (-1)^{|\sigma| + m} u_{\sigma_1 \varepsilon_1 f_1 + \ldots + \sigma_m \varepsilon_m f_m}.$$

Then we see that

$$D^m_{\vec{\varepsilon}} \Box_g u_{\varepsilon_1 f_1 + \ldots + \varepsilon_m f_m} = -m! q v_1 \cdots v_m + D^m_{\vec{\varepsilon}} \Box \mathcal{R},$$

where $\mathcal{R}$ is an error term (depends on ε_j) which we can control. Here the functions v_j solve

$$\begin{cases} \Box_g v_j = 0, & \text{in } [0,T] \times \Omega, \\ v_j = f_j, & \text{on } \Sigma \\ v_j = \partial_t v_j = 0, & \text{at } t = 0 \end{cases}$$

for $j = 1, \ldots, m$.

Let v_0 be an auxiliary solution to

$$\begin{cases} \Box_g v_0 = 0, \\ v_0 = \partial_t v_0 = 0, \quad t = T. \end{cases}$$

Integrating the differentiated equation against v_0, we get the identity

$$
\begin{aligned}
-\, m! \int_{[0,T]\times\Omega} q\, v_0\, v_1\, v_2 \cdots v_m \, dV_g
&= \int_{\Sigma} v_0\, D^m_{\vec{\varepsilon}} \Lambda(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m)\, dS \\
&\quad + \frac{1}{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m} \int_{[0,T]\times\Omega} v_0 \Box \widetilde{\mathcal{R}}\, dV_g .
\end{aligned}
$$

Let's make $v_1 \cdots v_m \approx \delta$!

How to choose $v_1, \dots, v_m$?

We choose $v_1, \dots, v_m$ to be Gaussian beams. Gaussian beams are constructed by considering a lightlike geodesic γ. If s is the geodesic parameter, let (s, y), $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, be Fermi coordinates in a neighbourhood of the graph Γ of γ. A Gaussian beam looks roughly like

$$e^{iy_1\tau - a\tau|y|^2}, \quad a > 0,\, \tau \gg 1.$$

Qualitatively: oscillation to direction transversal to γ and exponential concentration on γ.

So Gaussian beams are like wave-packets travelling along lightlike geodesics.

We will choose two distinct geodesics γ_1 and γ_2 which intersect in W and set v_1 and v_2 to be Gaussian beams related to these geodesics. To cancel the oscillation, we choose $v_3 = \overline{v_1}$ and $v_4 = \overline{v_2}$.

(Brief) Construction of Gaussian beams

Gaussian beams are constructed by using a WKB ansatz

$$v(s, y) = e^{i\tau\Theta(s,y)}a(s, y)$$

to approximatively solve the equation $\Box_g v = 0$ in the Fermi coordinates (s, y). We have

$$\Box_g(e^{i\tau\Theta}a) = e^{i\tau\Theta}\big(\tau^2 g(d\Theta, d\Theta) - 2i\tau g(d\Theta, da) + i\tau(\Box_g\Theta)a + \Box_g a\big).$$

We will choose a *phase function* Θ and an *amplitude function* a so that the right hand side is $\mathcal{O}(\tau^{-K})$ in $H^k([0, T] \times \Omega)$ for a given $K \in \mathbb{N}$.

Construction of phase function

Approximatively solve the eikonal equation

$$g(d\Theta, d\Theta) = 0 :$$

set $\Theta = \sum_{j=0}^{N} \Theta_j(s, y)$, where $\Theta_j(s, y)$ is a homogeneous polynomial of order j in $y \in \mathbb{R}^n$. We say that $g(d\Theta, d\Theta)$ vanishes to order N on Γ, or that $g(d\Theta, d\Theta) = 0$ is satisfied to order N on Γ, if

$$(\partial_y^\alpha g(d\Theta, d\Theta))(s, 0) = 0,$$

where α is any multi-index with $|\alpha| \leq N$.) We set

$$\Theta_0 = 0 \text{ and } \Theta_1 = y_1.$$

It follows that

$$g(d\Theta, d\Theta)(s, 0) = 0 \text{ and } (\partial_{y_l} g(d\Theta, d\Theta))(s, 0) = 0,$$

where $l = 1, \dots, n$. That is, the eikonal equation is satisfied to order 1 on Γ. This process can be extended up to higher orders.

Construction of amplitude

After finding an (approximative) solution Θ to the eikonal equation, we solve for a by inserting Θ into

$$-2i\tau g(d\Theta, da) + i\tau(\Box_g\Theta)a + \Box_g a = 0.$$

By assuming an expansion $a = \sum_{j=0}^{N} \tau^{-j} a_j$ we are led by equating the powers of τ to a family of $N+1$ equations

$$\begin{aligned} &-2ig(d\Theta, da_0) + i(\Box_g\Theta)a_0 = 0, \\ &-2ig(d\Theta, da_j) + i(\Box_g\Theta)a_j - \Box_g a_{j-1} = 0, \end{aligned}$$

$j = 1, \ldots, N$. We solve these equations recursively in j starting from a_0.

Choosing v_0

So far our integral identity looks like

$$
\begin{aligned}
&- m! \int_{[0,T]\times\Omega} q v_0 |v_1|^2 |v_2|^2 v_5 \cdots v_m \, dV_g \\
&\qquad = \int_{\Sigma} v_0 D^m_{\vec{\varepsilon}} \Lambda(\varepsilon_1 f_1 + \cdots + \varepsilon_m f_m) \, dS \\
&\qquad\qquad + \frac{1}{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m} \int_{[0,T]\times\Omega} v_0 \Box \widetilde{\mathcal{R}} \, dV_g .
\end{aligned}
$$

Now we choose v_0 so that it has vanishing Cauchy data at $\{t = T\}$ and so that v_0 is supported near the intersection of γ_1 and γ_2.

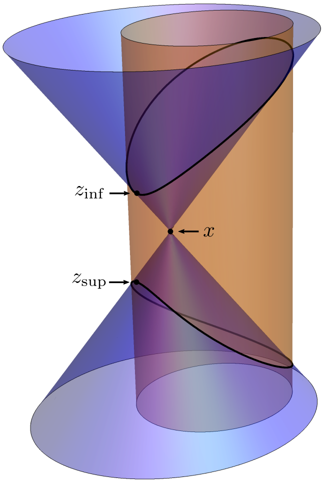
Boundary optimal geodesics

We say that a geodesic γ from p to q ($p \leq q$) is optimal, if the time-separation function τ satisfies $\tau(p,q) = 0$. Time-separation function is defined to be the supremum of lenghts of piecewise smooth causal paths from p to q.

Lemma (Boundary optimal geodesics)

Let (N, g) be globally hyperbolic, $N = \mathbb{R} \times M$. If $x \in I^+(\Sigma) \cap ([0,T] \times \Omega)$, there exists a future-directed optimal geodesic $\gamma : [0,1] \to [0,T] \times \Omega$ from Σ to x and the first intersection of γ and Σ is transverse. Similarly, if $x \in I^-(\Sigma) \cap ([0,T] \times \Omega)$, there exists a past-directed optimal geodesic $\gamma : [0,1] \to [0,T] \times \Omega$ from Σ to x and the first intersection of γ and Σ is transverse.

Dealing with non-convexity



We allow possibly non-convex domain. To deal with this, we introduced *boundary optimal geodesics*, which are lightlike geodesics that intersect the lateral boundary Σ at the earliest or latest time possible.
Such geodesics intersect the lateral boundary transversally.

Multiple intersections?

In general, two lightlike geodesics can intersect many times. Suppose we have chosen two lightlike geodesics γ_1 and γ_2 with their corresponding Gaussian beams. Let

$$\{x_1, \ldots, x_P\} = \gamma_1(\mathbb{R}) \cap \gamma_2(\mathbb{R}) \cap ([0, T] \times \Omega)$$

Due to exponential concentration of Gaussian beams, we then have that

$$-m! \int_{[0,T]\times\Omega} q v_0 |v_1|^2 |v_2|^2 v_5 \cdots v_m \, dV_g = \sum_{j=1}^{P} c_j v_0(x_j) q(x_j) + \text{error}$$

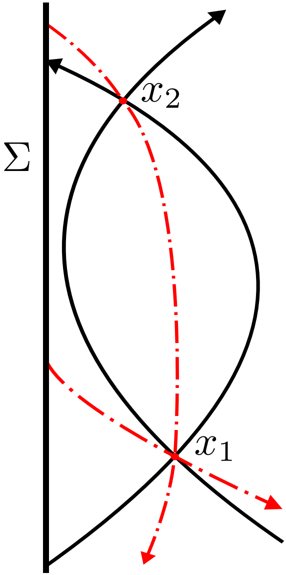
Separation of points

It is possible to choose a family $(v_0^{(k)})_{k=1}^P$ of P functions, satisfying the required conditions for v_0, with the property that the matrix

$$\mathcal{V} := \begin{pmatrix} v_0^{(1)}(x_1) & v_0^{(1)}(x_2) & \cdots & v_0^{(1)}(x_P) \\ v_0^{(2)}(x_1) & v_0^{(2)}(x_2) & \cdots & v_0^{(2)}(x_P) \\ \vdots & & \ddots & \vdots \\ v_0^{(P)}(x_1) & v_0^{(P)}(x_2) & \cdots & v_0^{(P)}(x_P) \end{pmatrix}$$

is invertible.

Separation of points



Separation of points can be done uniformly

Let $\overline{g}$ be a Riemannian metric on $\mathbb{R} \times M$.

Lemma (Separation filter)

Let $P \geq 1$ be an integer and let $\delta > 0$. Suppose $K \subset I^-(\Sigma) \cap I^+(\Sigma) \cap ([0,T] \times \Omega)$ is a compact set. There exists a finite collection $\mathcal{M} \subset C^\infty(\Sigma)$ of boundary values with the following properties: Assume that $x_1, \ldots, x_P \in K$ are any points such that $x_1 < x_2 < \cdots < x_P$ and $d_{\overline{g}}(x_k, x_l) > \delta$ for $x_k \neq x_l$, $k, l = 1, \ldots, P$. Then there are $f_1, \ldots, f_P \in \mathcal{M} \subset C^\infty(\Sigma)$ and corresponding solutions v_{f_k} of $\Box_g v_{f_k} = 0$ with vanishing Cauchy data at $t = T$, such that the separation matrix $(v_{f_i}(x_j))_{i,j=1}^P$ is invertible.

Intersections near the lateral boundary

Lemma

Let $\tau > 0$, $x \in \mathbb{R}^d_+$, $d \geq 2$, and assume $x = (x_1, \ldots, x_d)$, where $x_1 \geq 0$. Let $b : \mathbb{R}^d_+ \to \mathbb{R}$ be Lipschitz. Define a map $\Phi :]-\infty, 0] \to [1/2, 1]$ by $\Phi(s) := \frac{1}{\sqrt{\pi}} \int_s^\infty e^{-t^2} dt$. The following estimate

$$\left| b(x) - \frac{1}{\Phi(-\sqrt{\tau} x_1)} \left(\frac{\tau}{\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d \cap \{z_1 \geq 0\}} b(z) \mathrm{e}^{-\tau |z-x|^2} \mathrm{d}z \right| \leq 2 c_d \|b\|_{\mathrm{Lip}} \, \tau^{-1/2}$$

holds true for all $x \in \mathbb{R}^d \cap \{x_1 \geq 0\}$. In particular, the integral on the left converges uniformly to b as $\tau \to \infty$. Here $c_d = \Gamma(\frac{d+1}{2}) / \Gamma(\frac{d}{2})$.

Combining everything

One can show that everything so far can be done uniformly with respect to the intersection points $x_1, \ldots, x_P$ (using compactness of $[0, T] \times \Omega$).
So finally optimizing all parameters implicit in the equalities

$$-m! \int_{[0,T]\times\Omega} q v_0^{(i)} v_1 v_2 \cdots v_m dV_g \approx \sum_{j=1}^{P} c_j v_0^{(i)}(x_j) q(x_j) + \text{error}$$

$j = 1, \ldots, P$ recovers q up to a known error of Hölder type.

Reference

Lassas M, Liimatainen T, Potenciano-Machado L and Tyni T, Stability estimates for inverse problems for semi-linear wave equations on Lorentzian manifolds, arXiv:2106.122573, 2021


Thank you!