Recent results concerning “small” change in boundary conditions

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Some background – internal perturbations

ω_ϵ are measurable sets with (Lebesgue measure)

$|\omega_\epsilon| \to 0$ as $\epsilon \to 0$.

$$\gamma_\epsilon = \begin{cases} \gamma \text{ for } x \in \omega_\epsilon \\ 1 \text{ for } x \in \Omega \setminus \omega_\epsilon \end{cases}$$

$$\begin{cases} \nabla \cdot (\gamma_\epsilon \nabla u_\epsilon) = 0 & \text{in } \Omega \\ \frac{\partial u_\epsilon}{\partial n} = \psi & \text{on } \partial\Omega\ . \end{cases} \qquad \begin{cases} \Delta u_0 = 0 & \text{in } \Omega \\ \frac{\partial u_0}{\partial n} = \psi & \text{on } \partial\Omega\ . \end{cases}$$

There exists a probability measure and a polarization tensor valued function M – only depending on ω_ϵ and γ – so that

$$u_\epsilon(x)-u_0(x) = |\omega_\epsilon| \int_\Omega M(y)\nabla u_0(y)\cdot\nabla_y N(x,y)\, d\mu_y + o(|\omega_\epsilon|) \; .$$

$N(x,y)$ is the solution to

$$\Delta_y N(x,y) = -\delta_x \text{ in } \Omega \; , \; \frac{\partial}{\partial n_y} N(x,y) = -\frac{1}{|\partial\Omega|} \text{ on } \partial\Omega \; .$$

Remarks:

(1) this holds upon extraction of a subsequence.

(2) the asymptotics is **generally not** uniform in γ.

(3) the asymptotics holds **away** from ω_ϵ.

(4) The set ω_ϵ is supposed to be bounded away from $\partial\Omega$.

Uses:

this "representation" has been used for

(1) reconstruction of locations, when $\mu = \sum_{j=1}^{p} a_j \delta_{z_j}$,

(2) estimation of $|\omega_\epsilon|$ based on one or more applied currents ψ.

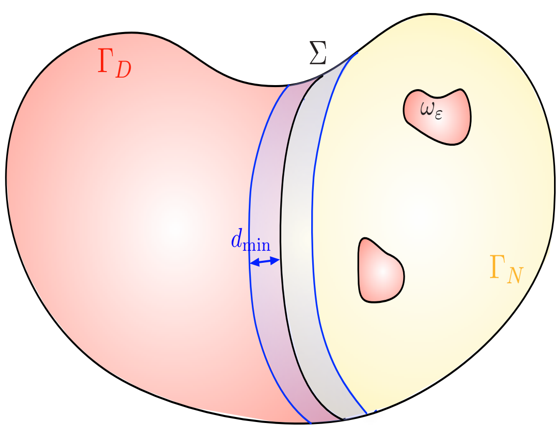
and, finally

(3) the fact that $\|u_\epsilon - u_0\|_{H^{1/2}(\partial\Omega)} \leq C|\omega_\epsilon|$, (**uniformly** in γ) when ω_ϵ is diametrically small, has been used to

establish near- invisibility estimates for approximate transformational cloaks.

Collaborators include: Yves Capdeboscq, Avner Friedman, Shari Moskow, Hoai-Minh Nguyen ...

But now let us consider the boundary condition perturbation problem:



The perturbation settings

$$\begin{cases} \Delta u_0 = f & \text{in } \Omega \\ \frac{\partial u_0}{\partial n} = 0 & \text{on } \Gamma_N \\ u_0 = 0 & \text{on } \Gamma_D \ . \end{cases}$$

$$\begin{cases} \Delta u_\epsilon = f & \text{in } \Omega \\ \frac{\partial u_\epsilon}{\partial n} = 0 & \text{on } \Gamma_N \setminus \overline{\omega_\epsilon} \\ u_\epsilon = 0 & \text{on } \Gamma_D \cup \omega_\epsilon \ . \end{cases} \qquad \begin{cases} \Delta u_\epsilon = f & \text{in } \Omega \\ \frac{\partial u_\epsilon}{\partial n} = 0 & \text{on } \Gamma_N \cup \omega_\epsilon \\ u_\epsilon = 0 & \text{on } \Gamma_D \setminus \overline{\omega_\epsilon} \ . \end{cases}$$

In the first case ω_ϵ is a subset of Γ_N, in the last case ω_ϵ is a subset of Γ_D.

GOAL: Find an asymptotic expression for $u_\epsilon(x) - u_0(x)$.

Assumptions: (1) $\omega_\epsilon \subset \partial\Omega$ consists of a finite number of connected, open Lipschitz subdomains, the closures of which do not intersect. (2) ω_ϵ lies either strictly inside Γ_N, or stricly inside Γ_D.

General Representation Formula

For $\omega_\epsilon \subset \Gamma_N$, when we impose $u_\epsilon = 0$ on ω_ϵ:

$$u_\epsilon(x) = u_0(x) - \mathrm{cap}_D(\omega_\epsilon) \int_{\partial\Omega} u_0(y) N(x,y) d\mu(y) + o(\mathrm{cap}_D(\omega_\epsilon))$$

for $x \in K \subset\subset \Omega$.

$N(x,y)$ is a fundamental solution associated with Δ:

$$-\Delta_y N(x,y) = \delta_x \text{ in } \Omega$$
$$\frac{\partial N}{\partial n_y} = 0 \text{ on } \Gamma_N \ , \quad N(x,\cdot) = 0 \text{ on } \Gamma_D \ .$$

The capacity $\mathrm{cap}_D(\omega_\epsilon)$ is defined by

$$\mathrm{cap}_D(\omega_\epsilon) = \min\left\{ \int_{\mathbb{R}^d} \left(|\nabla v|^2 + |v|^2 \right) \, dx \;:\; v \in H^1(\mathbb{R}^d) \;,\; v = 1 \text{ on } \omega_\epsilon \right\}$$

About $\mathrm{cap}_D(\omega_\epsilon)$

Define χ_ϵ by

$$\begin{aligned} &\Delta \chi_\epsilon = 0 \;\text{ in } \Omega \;, \\ &\frac{\partial \chi_\epsilon}{\partial n} = 0 \;\text{ on } \Gamma_N \setminus \overline{\omega_\epsilon} \;, \\ &\chi_\epsilon = 0 \;\text{ on } \Gamma_D \;\text{ and } \chi_\epsilon = 1 \;\text{ on } \omega_\epsilon \;. \end{aligned}$$

Then

$$c \int_\Omega |\nabla \chi_\epsilon|^2 \, dx \leq \mathrm{cap}_D(\omega_\epsilon) \leq C \int_\Omega |\nabla \chi_\epsilon|^2 \, dx$$

$$\text{and} \quad \int_\Omega |\chi_\epsilon|^2 \, dx \leq C \; \mathrm{cap}_D(\omega_\epsilon)^{3/2}$$

We calculate

$$\frac{1}{\mathrm{cap}_D(\omega_\epsilon)} \int_{\partial\Omega} \frac{\partial \chi_\epsilon}{\partial n} \chi_\epsilon \, \phi \qquad = \frac{1}{\mathrm{cap}_D(\omega_\epsilon)} \int_\Omega \nabla \chi_\epsilon \cdot \nabla(\chi_\epsilon \phi)$$

$$= \frac{1}{\mathrm{cap}_D(\omega_\epsilon)} \int_\Omega |\nabla \chi_\epsilon|^2 \phi + O(\|\phi\|_{C^1} \mathrm{cap}_D(\omega_\epsilon)^{1/4})$$

and so

$$\frac{1}{\mathrm{cap}_D(\omega_\epsilon)} \frac{\partial \chi_\epsilon}{\partial n} \chi_\epsilon \ \text{ converges weak* in } (C^1(\partial\Omega))' \text{ to some } \mu$$

(after extraction of a sub-sequence)

furthermore $\quad |\mu(\phi)| \leq C\|\phi\|_{C^0(\partial\Omega)}$ for all $\phi \in C^1(\partial\Omega)$,

in other words μ is a positive Radon measure

$$\mu(\phi) = \int_{\partial\Omega} \phi d\mu$$

Note: the support of μ lies inside any compact set, which contains all ω_ϵ (from a certain point in the sub-sequence).

Let $r_\epsilon = u_\epsilon - u_0$. This satisfies the estimates

$$\|r_\epsilon\|_{H^1(\Omega)} \leq C \operatorname{cap}_D(\omega_\epsilon)^{1/2} \quad \text{and} \quad \|r_\epsilon\|_{L^2(\Omega)} \leq C \operatorname{cap}_D(\omega_\epsilon)^{3/4} \ .$$

Suppose ψ vanishes on Γ_D, then

$$\begin{aligned}
\int_{\partial\Omega} \frac{\partial r_\epsilon}{\partial n}\, \psi &= \int_\Omega \nabla r_\epsilon \cdot \nabla(\chi_\epsilon \psi) \\
&= \int_\Omega \nabla r_\epsilon \cdot \nabla \chi_\epsilon\, \psi + O(\operatorname{cap}_D(\omega_\epsilon)^{5/4}) \\
&= \int_{\partial\Omega} r_\epsilon \frac{\partial \chi_\epsilon}{\partial n}\, \psi + O(\operatorname{cap}_D(\omega_\epsilon)^{5/4}) \\
&= \int_{\partial\Omega} r_\epsilon \frac{\partial \chi_\epsilon}{\partial n} \chi_\epsilon\, \psi + O(\operatorname{cap}_D(\omega_\epsilon)^{5/4}) \\
&= -\int_{\partial\Omega} u_0 \frac{\partial \chi_\epsilon}{\partial n} \chi_\epsilon\, \psi + O(\operatorname{cap}_D(\omega_\epsilon)^{5/4}) \ .
\end{aligned}$$

With $\psi = N(x, \cdot)$:

$$\int_{\partial\Omega} \frac{\partial r_\epsilon}{\partial n_y} N(x,y)\, ds_y = -\int_{\partial\Omega} u_0(y) \frac{\partial \chi_\epsilon}{\partial n_y} \chi_\epsilon\, N(x,y)\, ds_y + O(\mathrm{cap}_D(\omega_\epsilon)^{5/4}) \,,$$

and so

$$\lim \frac{1}{\mathrm{cap}_D(\omega_\epsilon)} \int_{\partial\Omega} \frac{\partial r_\epsilon}{\partial n_y} N(x,y)\, ds_y = -\int_{\partial\Omega} u_0(y)\, N(x,y)\, d\mu_y$$

We also calculate

$$\begin{aligned} u_\epsilon(x) - u_0(x) &= r_\epsilon(x) = -\int_\Omega r_\epsilon(y) \Delta_y N(x,y)\ dy \\ &= \int_\Omega \nabla r_\epsilon \cdot \nabla_y N(x,y)\ dy \\ &= \int_{\partial\Omega} \frac{\partial r_\epsilon}{\partial n} N(x,y)\, ds_y\ . \end{aligned}$$

Altogether

$$\lim \frac{1}{\mathrm{cap}_D(\omega_\epsilon)} (u_\epsilon(x) - u_0(x)) = -\int_{\partial\Omega} u_0(y)\, N(x,y)\, d\mu_y \ ,$$

or

$$u_\epsilon(x) = u_0(x) - \mathrm{cap}_D(\omega_\epsilon) \int_{\partial\Omega} u_0(y)\, N(x,y)\, d\mu_y + o(\mathrm{cap}_D(\omega_\epsilon)) \ .$$

Immediate application:

$$\begin{aligned} \int_\Omega u_\epsilon\, f \ dx \quad &= \quad \int_\Omega u_0\, f \ dx \\ &\qquad -\mathrm{cap}_D(\omega_\epsilon) \int_{\partial\Omega} u_0(y) \ \int_\Omega N(x,y)\, f(x)\, dx\, d\mu_y \\ &= \quad \int_\Omega u_0\, f \ dx \\ &\qquad -\mathrm{cap}_D(\omega_\epsilon) \int_{\partial\Omega} u_0(y)^2\, d\mu_y \leq \int_\Omega u_0\, f \ dx \ , \end{aligned}$$

in other words: the compliance asymptotically (strictly) decreases by introducing a small “clamped” area inside the “free” boundary – by how much depends on the size and shape of ω_ϵ and on u_0^2.

We have a similar asymptotic formula for the insertion of homogeneous Neumann boundary conditions on ω_ϵ inside Γ_D

$$u_\epsilon(x) = u_0(x) + e(\omega_\epsilon) \int_{\partial\Omega} \frac{\partial u_0}{\partial n}(y) \frac{\partial N}{\partial n_y}(x, y)\, d\nu_y + o(e(\omega_\epsilon)) \ .$$

$e(\omega_\epsilon)$ (you may call it the Neumann capacity) and the Radon measure ν are defined in a way that parallels the case before:

Let $\kappa \in C_c^\infty(\mathbb{R}^d)$ with $\kappa = \pm 1$ on ω_ϵ. ω_ϵ has only finitely connected components, and so there are only finitely many choices of ± 1. Define z_κ by

$$\begin{aligned} -\Delta z_\kappa + z_\kappa &= 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\omega_\epsilon} \ , \\ \frac{\partial z_\kappa}{\partial n} &= \kappa \quad \text{on } \omega_\epsilon \ . \end{aligned}$$

Then

$$e(\omega_\epsilon) = \max_\kappa \left\{ \int_{\mathbb{R}^d \setminus \omega_\epsilon} (|\nabla z_\kappa|^2 + z_\kappa^2) \right\} \ .$$

Define ζ_ϵ by

$$\begin{aligned} &\Delta \zeta_\epsilon = 0 \quad \text{in } \Omega \ , \\ &\frac{\partial \zeta_\epsilon}{\partial n} = 0 \quad \text{on } \Gamma_N \text{ and } \frac{\partial \zeta_\epsilon}{\partial n} = 1 \quad \text{on } \omega_\epsilon \ , \\ &\zeta_\epsilon = 0 \quad \text{on } \Gamma_D \setminus \overline{\omega_\epsilon} \ . \end{aligned}$$

Then

$$c \int_\Omega |\nabla \zeta_\epsilon|^2 \ dx \le e(\omega_\epsilon) \le C \int_\Omega |\nabla \zeta_\epsilon|^2 \ dx$$

and $\quad \int_\Omega |\zeta_\epsilon|^2 \; dx \le Ce(\omega_\epsilon)^{3/2}$

Similar to before the positive Radon measure ν is obtained as the weak* limit of

$$\frac{1}{e(\omega_\epsilon)} \frac{\partial \zeta_\epsilon}{\partial n} \zeta_\epsilon \; .$$

Immediate application:

$$\begin{aligned}
\int_\Omega u_\epsilon \, f \; dx \quad &= \quad \int_\Omega u_0 \, f \; dx \\
&\qquad + e(\omega_\epsilon) \int_{\partial\Omega} \frac{\partial u_0}{\partial n}(y) \, \frac{\partial}{\partial n_y} \int_\Omega N(x,y) \, f(x) \, dx \, d\nu_y \\
&= \quad \int_\Omega u_0 \, f \; dx \\
&\qquad + e(\omega_\epsilon) \int_{\partial\Omega} \left(\frac{\partial u_0}{\partial n}(y) \right)^2 \, d\nu_y \ge \int_\Omega u_0 \, f \; dx \; ,
\end{aligned}$$

in other words: the compliance asymptotically (strictly) increases by introducing a small “free area” inside the “clamped” boundary – by how much depends on the size and shape of ω_ϵ and on $\left(\frac{\partial u_0}{\partial n}\right)^2$.

Some concrete examples:

Suppose ω_ϵ is a “surfacic” ball

$$\omega_\epsilon = \{y \ : \ |y - y_0| < \epsilon\} \cap \partial\Omega \quad \text{for some } y_0 \in \partial\Omega \ ,$$

Then

$$\text{cap}_D(\omega_\epsilon) = \begin{cases} O(\frac{1}{|\log \epsilon|}) & \text{for } d = 2 \\ O(\epsilon) & \text{for } d = 3 \end{cases} .$$

and

$$e(\omega_\epsilon) = O(\epsilon^d) \ \ d = 2, 3 \ .$$

In this case the measures μ and ν must be point masses (with support at y_0)

We now have the asymptotic formulas

(1) For insertion of a homogeneous Dirichlet boundary condition on $\omega_\epsilon \subset \Gamma_N$:

$$u_\epsilon(x) = u_0(x) - \frac{\pi}{|\log(\epsilon)|} u_0(y_0) N(x, y_0) + o(\frac{1}{|\log(\epsilon)|}) \quad \text{for } d = 2 \ ,$$

and

$$u_\epsilon(x) = u_0(x) - 4\epsilon u_0(y_0) N(x, y_0) + o(\epsilon) \quad \text{for } d = 3 \ .$$

(2) For insertion of a homogeneous Neumann boundary condition on $\omega_\epsilon \subset \Gamma_D$:

$$u_\epsilon(x) = u_0(x) + a_d \epsilon^d \frac{\partial u_0}{\partial n}(y_0) \, \frac{\partial N}{\partial n_y}(x, y_0) + o(\epsilon^d) \ .$$

with $a_2 = \frac{\pi}{2}$, and $a_3 = \frac{4}{3}$.