Imaging with nonlinear and fractionally damped waves

Barbara Kaltenbacher

Alpen-Adria-Universität Klagenfurt

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joint work with

Bill Rundell, Texas A&M University

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Modeling - Analysis - Optimization
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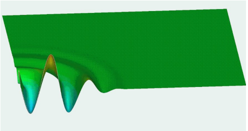
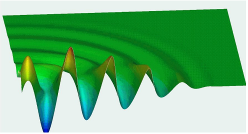
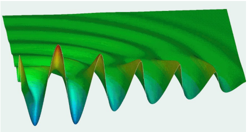
UNIVERSITÄT KLAGENFURT

Outline

- models of nonlinear acoustics
- models of fractional damping
- photoacoustic tomography PAT with fractional damping
- nonlinearity parameter imaging

models of nonlinear acoustics

Nonlinear Acoustic Wave Propagation

nonlinear wave propagation:

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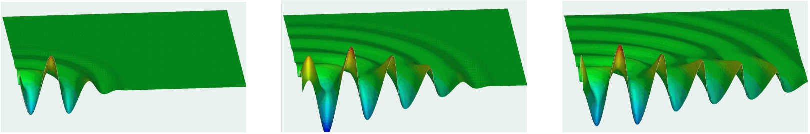

nonlinear wave propagation:

sound speed depends on (signed) amplitude ⇒ sawtooth profile

Physical Principles

main physical quantities:

- acoustic particle velocity $\vec{v}$;
- acoustic pressure p;
- mass density ϱ;

decomposition into mean and fluctuating part:

$$\vec{v} = \vec{v}_0 + \vec{v}_\sim = \vec{v}\,, \qquad p = p_0 + p_\sim\,, \qquad \varrho = \varrho_0 + \varrho_\sim$$

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governing equations: with phys. const. $\bar{\mu} = \left(\frac{4\mu_V}{3} + \zeta_V\right)$, $\tilde{\mu} := \kappa\left(\frac{1}{c_V} - \frac{1}{c_p}\right)$

- Navier Stokes equation (balance of momentum) with $\nabla \times \vec{v} = 0$

$$\varrho\Big(\vec{v}_t + \nabla(\vec{v}\cdot\vec{v})\Big) + \nabla p = \bar{\mu}\Delta\vec{v}$$

- equation of continuity (balance of mass)

$$\nabla \cdot (\varrho\vec{v}) = -\varrho_t$$

- equation of state (material law) $\frac{B}{A}\ldots$ nonlinearity parameter

$$\frac{\varrho_\sim}{\varrho_0} = \frac{p_\sim}{p_0} - \frac{B}{2A}\left(\frac{p_\sim}{p_0}\right)^2 - \tilde{\mu}\frac{p_{\sim t}}{p_0^2}$$

Derivation of Wave Equation

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- fluctuating quantities:

$$\nabla p = \nabla p_\sim \,, \qquad \varrho_t = \varrho_{\sim t}$$

Derivation of Wave Equation

$$\varrho_0 \vec{v}_t + \nabla p_\sim = 0$$

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\begin{aligned}
{\color{red}-\nabla\cdot} \qquad & \varrho_0 \vec{v}_t + \nabla p_\sim = 0 \\
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$$\frac{1}{c^2} p_{\sim tt} - \Delta p_\sim = 0$$

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Classical Models of Nonlinear Acoustics I

- **Kuznetsov's equation** [Lesser & Seebass 1968, Kuznetsov 1971]

$$p_{\sim tt} - c^2\Delta p_\sim - b\Delta p_{\sim t} = -\left(\frac{B}{2A\varrho_0 c^2} p_\sim^2 + \varrho_0 |\vec{v}|^2\right)_{tt}$$

where $\varrho_0 \vec{v}_t = -\nabla p$
for the **particle velocity** $\vec{v}$ and the **pressure** p, i.e.,

$$\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = -\left(\frac{B}{2A\,c^2}(\psi_t)^2 + |\nabla\psi|^2\right)_t$$

since $\nabla \times \vec{v} = 0$ hence $\vec{v} = -\nabla\psi$ for a **velocity potential** ψ

- **Westervelt equation** [Westervelt 1963]

$$p_{\sim tt} - c^2\Delta p_\sim - b\Delta p_{\sim t} = -\frac{1}{\varrho_0 c^2}\left(1 + \frac{B}{2A}\right) p^2_{\sim tt}$$

via $\varrho_0^2 |\vec{v}|^2 \approx \frac{1}{c^2}(p_{\sim t})^2$.

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via $\varrho_0^2 |\vec{v}|^2 \approx \frac{1}{c^2}(p_{\sim t})^2$. $\frac{B}{2A}$ ... nonlinearity parameter.

The Westervelt equation

with $\kappa := \frac{1+\frac{B}{2A}}{\varrho_0 c^2}$, $u = p_\sim$

$$u_{tt} - c^2 \Delta u - b \Delta u_t = -\kappa (u^2)_{tt}$$

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This also illustrates state dependence of the effective wave speed:

$$u_{tt} - \tilde{c}^2\Delta u - \tilde{b}(u)\Delta u_t = f(u)$$

with $\tilde{c}(u) = \frac{c}{\sqrt{1-2\kappa u}}$, $\tilde{b}(u) = \frac{b}{1-2\kappa u}$, $f(u) = \frac{2\kappa(u_t)^2}{1-2\kappa u}$
as long as $2\kappa u < 1$ (otherwise the model loses its validity)

fractional damping models in ultrasonics

Fractional Models of (Linear) Viscoelasticity

- equation of motion (resulting from balance of forces)

$$\varrho \vec{u}_{tt} = \mathrm{div}\sigma + \vec{f}$$

- strain as symmetric gradient of displacements:

$$\epsilon = \frac{1}{2}(\nabla \vec{u} + (\nabla \vec{u})^T).$$

- constitutive model: stress-strain relation

$\vec{u}$. . . displacements
σ. . . stress tensor
ϵ. . . strain tensor
ϱ. . . mass density

Fractional Models of (Linear) Viscoelasticity 1-d setting

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$$\epsilon = u_x .$$

- constitutive model: stress-strain relation:

$$\begin{aligned} \text{Hooke's law (pure elasticity):} \quad & \sigma = b_0 \epsilon \\ \text{Newton model:} \quad & \sigma = b_1 \epsilon_t \\ \text{Kelvin-Voigt model:} \quad & \sigma = b_0 \epsilon + b_1 \epsilon_t \\ \text{Maxwell model:} \quad & \sigma + a_1 \sigma_t = b_0 \epsilon \\ \text{Zener model:} \quad & \sigma + a_1 \sigma_t = b_0 \epsilon + b_1 \epsilon_t \end{aligned}$$

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- constitutive model: stress-strain relation:

fractional Newton model: $\sigma = b_1 \partial_t^\beta \epsilon$

fractional Kelvin-Voigt model: $\sigma = b_0 \epsilon + b_1 \partial_t^\beta \epsilon$

fractional Maxwell model: $\sigma + a_1 \partial_t^\alpha \sigma = b_0 \epsilon$

fractional Zener model: $\sigma + a_1 \partial_t^\alpha \sigma = b_0 \epsilon + b_1 \partial_t^\beta \epsilon$

general model class: $\sum_{n=0}^{N} a_n \partial_t^{\alpha_n} \sigma = \sum_{m=0}^{M} b_m \partial_t^{\beta_m} \epsilon$

[Caputo 1967, Atanackovic, Pilipović, Stanković, Zorica 2014]

Fractional Models of (Linear) Acoustics

balance of momentum $$\varrho_0 \vec{v}_t = -\nabla p + \vec{f}$$

balance of mass $$\varrho \nabla \cdot \vec{v} = -\varrho_t$$

equation of state $$\frac{\varrho_\sim}{\varrho_0} = \frac{p_\sim}{p_0}$$

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insert constitutive equations into combination of balance laws
$\rightsquigarrow$ fractional acoustic wave equations [Holm et al 2003 ff, Szabo 1994]:

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$\rightsquigarrow$ fractional acoustic wave equations [Holm et al 2003 ff, Szabo 1994]:

- Caputo-Wismer-Kelvin wave equation (fractional Kelvin-Voigt):
$$p_{tt} - b_0 \triangle p - b_1 \partial_t^{\beta} \triangle p = \tilde{f} \,,$$
- modified Szabo wave equation (fractional Maxwell):
$$p_{tt} - a_1 \partial_t^{2+\alpha} p - b_0 \triangle p = \tilde{f} \,,$$
- fractional Zener wave equation:
$$p_{tt} - a_1 \partial_t^{2+\alpha} p - b_0 \triangle p + b_1 \partial_t^{\beta} \triangle p = \tilde{f} \,,$$
- general fractional model:
$$\sum_{n=0}^{N} a_n \partial_t^{2+\alpha_n} p - \sum_{m=0}^{M} b_m \partial_t^{\beta_m} \triangle p = \tilde{f} \,.$$

Fractional derivatives

Abel fractional integral operator

$$I_a^{\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(s)}{(t-s)^{1-\gamma}}\, ds$$

Then a fractional (time) derivative can be defined by either

$${}_a^R D_t^{\alpha} f = \frac{d}{dt} I_a^{1-\alpha} f \quad \text{Riemann-Liouville derivative}$$

or

$${}_a^C D_t^{\alpha} f = I_a^{1-\alpha} \frac{df}{ds} \quad \text{Djrbashian-Caputo derivative}$$

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- R-L is defined on a larger function space, but derivative of constant is nonzero; singularity at initial time a
- D-C maps constants to zero $\rightsquigarrow$ appropriate for prescribing initial values

Nonlocal and causal character of these derivatives provides them with a "memory" $\rightsquigarrow$ initial values are tied to later values and can therfore be better reconstructed backwards in time.

PAT with fractional attenuation

Motivation

- attenuation of ultrasound in human tissue follows a power law frequency dependence ω^α
  $\rightsquigarrow$ fractional derivative ∂_t^α term in time domain
- PAT/TAT (sub)problem: Reconstruct initial pressure from observations of pressure at some transducer array over time
- only mildly ill-posed without attenuation
- severely ill-posed in with integer (1st) order damping

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? Uniqueness and reconstruction for PAT/TAT with fractional attenuation

? Dependence of instability on fractional differentition order

The inverse problem of PAT and TAT

Identify $u_0(x)$ in

$$u_{tt} + c^2 \mathcal{A}u + Du = 0 \text{ in } \Omega \times (0, T)$$
$$u(0) = u_0, \quad u_t(0) = 0 \text{ in } \Omega$$

where $\mathcal{A}u = -\triangle$ with homogeneous Dirichlet boundary conditions from observations

$$g = u \quad \text{on } \Sigma \times (0, T)$$

$\Sigma \subset \overline{\Omega}$... transducer array (surface or collection of discrete points)

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Caputo-Wismer-Kelvin:

$$D = b\mathcal{A}\partial_t^\beta \quad \text{with } \beta \in [0,1], \quad b \geq 0$$

fractional Zener:

$$D = a\partial_t^{2+\alpha} + b\mathcal{A}\partial_t^\beta \quad \text{with } a > 0,\ b \geq ac^2,\ 1 \geq \beta \geq \alpha > 0,$$

space fractional Chen-Holm:

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FZ fractional Zener:

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Uniqueness

Linear independence assumption:
For each eigenvalue λ of $\mathcal{A}$ with eigenfunctions $(\varphi_k)_{k\in K^\lambda}$, the restrictions of the eigenfunctions to the observation manifold are linear independent: For any coefficient set $(b_k)_{k\in K^\lambda}$

$$\left(\sum_{k\in K^\lambda} b_k\varphi_k(x) = 0 \text{ for all } x \in \Sigma\right) \implies \Big(b_k = 0 \text{ for all } k \in K^\lambda\Big) .$$

Theorem

Suppose the domain Ω and the operator $\mathcal{A}$ are known. Then under the linear independence assumption we can uniquely recover the initial value $u_0(x)$ from time trace measurements g on Σ.

Some remarks

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- Uniqueness of $c_0(x)$ from the same observations can be shown by Sturm-Liouville theory in 1-d.
- tools of proof:
  separation of variables (solution representation),
  analysis in Laplace domain (location of poles),
  uniqueness of eigenvalues from poles.

[BK&Rundell. Inverse Problems, 37(4):045002]

Separation of variables and relaxation equation

$(\lambda_n, \varphi_n)\ldots$ eigensystem of $\mathcal{A}$: $\quad u(x,t) = \sum_{j=1}^{\infty} u_j(t)\varphi_j(x)\,,$
where for each $j \in \mathbb{N}$, u_j solves

$$\widetilde{D}_{\lambda_j} u_j = 0 \text{ on } (0,T) \qquad u_j(0) = \langle u_0, \varphi_j\rangle, \quad u_j'(0) = 0$$

with

$$\widetilde{D}_\lambda = \partial_t^2 + c^2\lambda + \begin{cases} b\lambda^{\tilde\beta}\partial_t^\beta & \text{for C...H} \\ a\partial_t^{2+\alpha} + b\lambda\partial_t^\beta & \text{for FZ} \end{cases}$$

With relaxation functions w_λ solving the relaxation equations

$$\widetilde{D}_\lambda w_\lambda = 0 \text{ on } (0,T)\,, \quad w_\lambda(0) = 1\,,\ w_\lambda'(0) = 0 \text{ (if } \alpha > 0:\ w_\lambda''(0) = 0)$$

$$u(x,t) = \sum_{j=1}^{\infty} w_{\lambda_j}(t)\,\langle u_0, \varphi_j\rangle\,\varphi_j(x).$$

Uniqueness of eigenvalues from poles of relaxation functions

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Laplace transform $\Rightarrow \hat{w}_\lambda(s) = \frac{(\omega_\lambda(s) - c^2\lambda)/s}{\omega_\lambda(s)}$ where

$$\omega_\lambda = s^2 + c^2\lambda + \begin{cases} b\lambda^{\tilde{\beta}} s^\beta & \text{for } \mathrm{C}\dots\mathrm{H} \\ as^{2+\alpha} + b\lambda s^\beta & \text{for } \mathrm{FZ} \end{cases}$$

Lemma

The poles of $\hat{w}$ differ for different λ.[1]

[1](except for $p_0^{\mathrm{FZ}} = -1/a$ in case $\beta = \alpha = 1$)

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The residues of the poles of $\hat{w}$ do no vanish.[1]

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Proof of Uniqueness Theorem

(A) The poles of $\hat{w}$ differ for different λ.

(B) The residues of the poles of $\hat{w}$ do no vanish.

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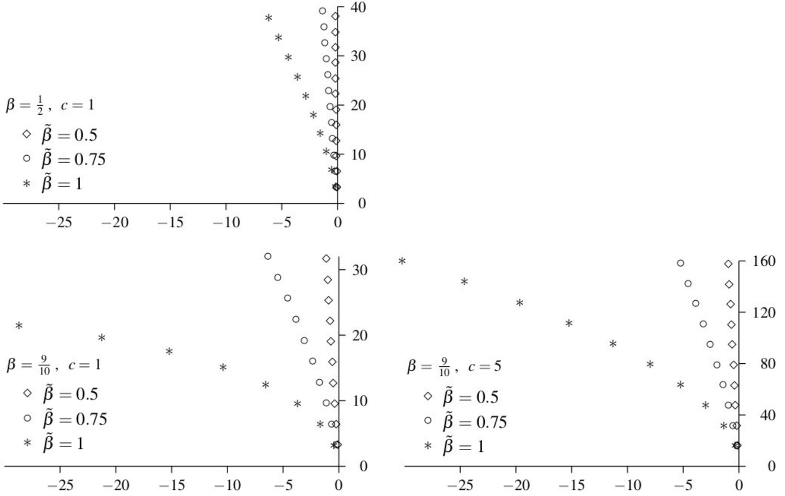
Location of poles

$\mathrm{C}\ldots\mathrm{H}$ Caputo-Wismer-Kelvin / space fractional Chen-Holm:

$$u_{tt} - c^2 \triangle u + b(-\triangle)^{\tilde{\beta}} \partial_t^{\beta} u = 0$$



Roots of $\omega_{\mathrm{C}\ldots\mathrm{H}}(s)$ **for various** β, $\tilde{\beta}$, c **values.**

Reconstructions I

$\Omega = (0, 1)$

$g(t) = \sum_{j=1}^{\infty} w_{\lambda_j}(t) \langle u_0, \varphi_j \rangle \varphi_j(x_0), \; t \in (0, T)$

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where

$$\mathbb{W} = (w_{\lambda_j}(t_i))_{i=1,\dots,M,\, j=1,\dots N},$$

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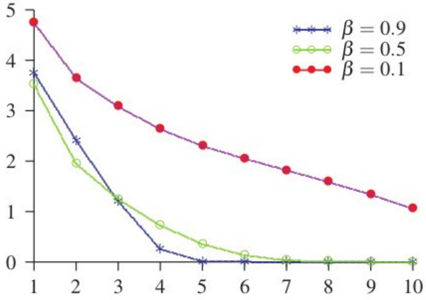

Singular values of $\mathbb{W}$ with different values of β; here $\tilde{\beta} = 1$

Reconstructions II

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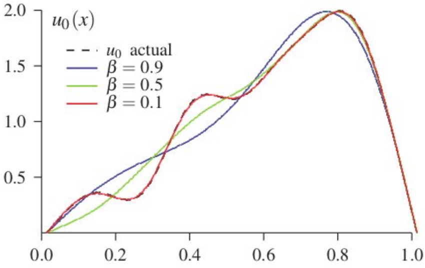

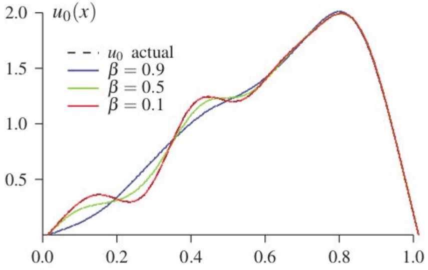

Recovery of $u_0(x)$ from $g_\beta(t)$. Left: with 1% noise; Right: with 0.1% noise

nonlinearity parameter imaging

The inverse problem

Identify $\kappa(x)$ in

$$\left(u - \kappa(x) u^2\right)_{tt} - c_0^2 \triangle u + Du = r \quad \text{in } \Omega \times (0, T)$$

$$u = 0 \text{ on } \partial\Omega \times (0, T), \quad u(0) = 0, \quad u_t(0) = 0 \quad \text{in } \Omega$$

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$$g = u \quad \text{on } \Sigma \times (0, T)$$

$\Sigma \subset \overline{\Omega} \ldots$ transducer array (surface or collection of discrete points)

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fractional damping

Caputo-Wismer-Kelvin:

$$D=-b\Delta\partial_t^\beta \quad \text{with } \beta\in[0,1], \quad b\geq 0$$

fractional Zener:

$$D=a\partial_t^{2+\alpha}-b\Delta\partial_t^\beta \quad \text{with } a>0,\ b\geq ac^2,\ 1\geq\beta\geq\alpha>0,$$

space fractional Chen-Holm:

$$D=b(-\Delta)^{\tilde\beta}\partial_t \quad \text{with } \tilde\beta\in[0,1],\ b\geq 0,$$

Challenges

- model equation is nonlinear;
  nonlinearity occurs in highest order term;
- unknown coefficient $\kappa(x)$ appears in this nonlinear term
- κ is spatially varying whereas the data $g(t)$ is in the "orthogonal" time direction;
  This is well known to lead to severe ill-conditioning of the inversion of the map F from data to unknown.

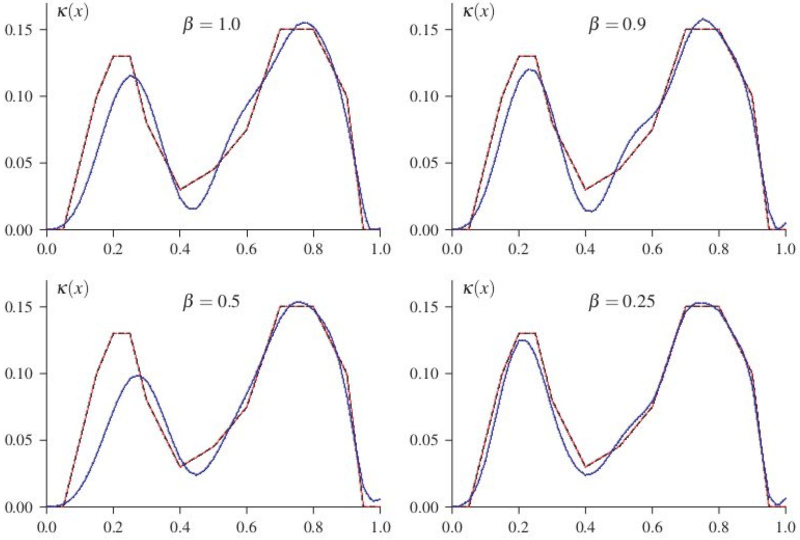
Results

- Well-definedness and Fréchet differentiability of forward operator $F : \kappa \mapsto u|_\Sigma$
- uniqueness of linearized problem under linear independence assumption
- reconstructions by Newton's method

[BK&Rundell IPI 2021, Math.Comp. 2021]
see also arXiv:2103.08965 [math.AP], arXiv:2102.07608 [math.AP]

Reconstructions of $\kappa(x)$

Caputo-Wismer-Kelvin: $D = -b\Delta\partial_t^{\beta}$, 0.1% noise



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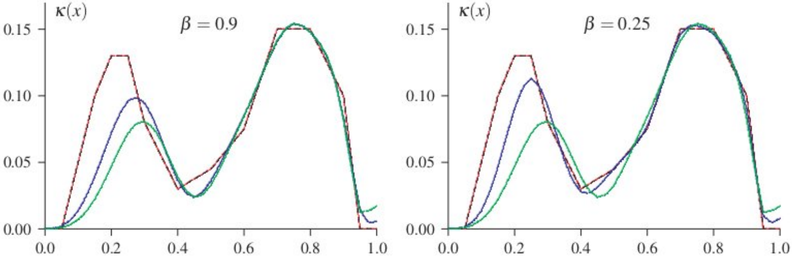
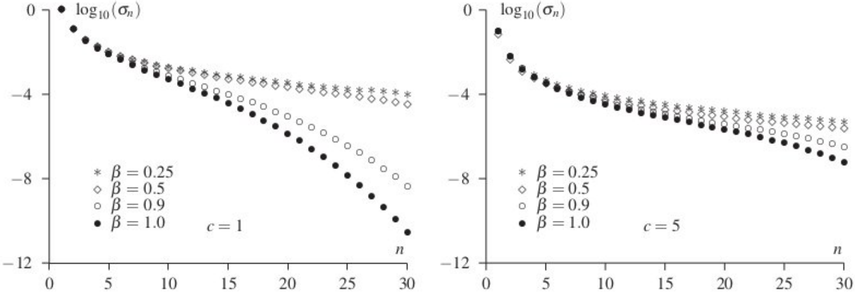


0.5% (blue) and 1% (green) noise

Singular values of linearized forward operator



Thank you for your attention!