Inverse problems for fractional parabolic equations with power type nonlinearities

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Outline

Uniqueness results for:

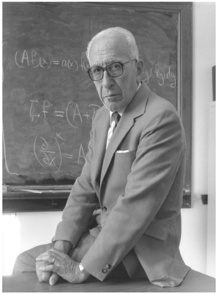
- Classical Calderón problem
- Two fractional Calderón problems
- Two inverse problems for fractional parabolic equations with power type nonlinearities

Notations

- $n \geq 2$: space dimension
- $0 < s < 1$: fractional power
- Ω : a bounded domain with smooth boundary $\partial\Omega$
- $\Omega_e := \mathbb{R}^n \setminus \bar{\Omega}$
- $\langle \cdot, \cdot \rangle$: distributional pairing
- $H^s(U)$: the Sobolev space $W^{s,2}(U)$

1. Classical Calderón Problem

Question: Can we determine the electrical conductivity of a medium by making voltage and current measurements at its boundary?



Alberto P. Calderón

Mathematical formulation:

- A conductor fills a bounded domain Ω.
- $\gamma(x)$ denotes the electrical conductivity at x.
- We apply a voltage f on $\partial\Omega$.
- $u(x)$ denotes the induced voltage at x.
- Ohm's Law: current at x is $-\gamma(x)\nabla u(x)$.
- In the absence of sinks or sources of current, u satisfies

$$\mathrm{div}(\gamma\nabla u) = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = f.$$

1. Classical Calderón Problem

- The Dirichlet-to-Neumann (voltage-to-current) map is

$$\Lambda_\gamma : f \to (\gamma \frac{\partial u}{\partial \nu})|_{\partial\Omega}$$

where ν is the unit outer normal on $\partial\Omega$.

- Inverse problem: Does $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ imply $\gamma_1 = \gamma_2$?
- Fundamental uniqueness theorem:

(Sylvester-Uhlmann, 87)

Suppose $n \geq 3$. Let $\gamma_{1,2} \in C^2(\bar{\Omega})$ be strictly positive. If

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then $\gamma_1 = \gamma_2$ in Ω.

1. Classical Calderón Problem

Reduction to Schrödinger equation:

- Use the substitution $q = (\Delta\sqrt{\gamma})/\sqrt{\gamma}$ to convert the Dirichlet problem into

$$(-\Delta + q)u = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = f.$$

- It suffices to determine q from the associated DN map

$$\Lambda_q : f \to \frac{\partial u}{\partial \nu}|_{\partial\Omega}.$$

Main ingredients of the proof:

- Integral identity for Dirichlet-to-Neumann maps
- Complex geometrical optics solutions

1. Classical Calderón Problem

Integral identity:

- Let u_j $(j = 1, 2)$ be the solution of

$$(-\Delta + q_j)u = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = f_j.$$

  We can show that

$$\langle(\Lambda_{q_1} - \Lambda_{q_2})f_1, f_2\rangle = \int_\Omega (q_1 - q_2)u_1 u_2.$$

- Hence $\Lambda_{q_1} = \Lambda_{q_2}$ implies that

$$\int_\Omega (q_1 - q_2)u_1 u_2 = 0.$$

1. Classical Calderón Problem

CGO solutions:

- Consider solutions of the form

$$u_j = e^{x\cdot\rho_j}(1 + \psi_{q_j}(x, \rho_j))$$

where

$$\rho_{1,2} = \pm\frac{\eta}{2} + i\frac{k \pm l}{2},$$

$\eta, k, l \in \mathbb{R}^n$ ($n \geq 3$) satisfy

$$\eta \cdot k = \eta \cdot l = k \cdot l = 0, \qquad |\eta|^2 = |k|^2 + |l|^2$$

and $\psi_{q_j} \to 0$ as $|\rho_j| \to \infty$.

- Let $|l| \to \infty$ in the integral identity to obtain

$$\int_\Omega e^{ix\cdot k}(q_1 - q_2) = 0.$$

1. Classical Calderón Problem

Remarks:

- The uniqueness theorem holds true for $n = 2$ (Nachman, 96; Brown-Uhlmann, 97; Astala-Päivärinta, 06; Bukhgeim, 08) but a different proof is required.
- For $\Gamma_1, \Gamma_2 \subset \partial\Omega$, does the partial DN map

$$\Lambda_\gamma^{\Gamma_1,\Gamma_2}(f) := (\gamma \frac{\partial u}{\partial \nu})|_{\Gamma_2}, \quad \mathrm{supp}\, f \subset \Gamma_1.$$

  determine γ in Ω? This problem is open in general but much progress has been made so far (Bukhgeim-Uhlmann, 02; Ammari-Uhlmann, 04; Kenig-Sjöstrand-Uhlmann, 07; Kenig-Salo, 14; Krupchyk-Uhlmann, 16).

2. Fractional Calderón Problem

- Nonlocal operators (e.g. the fractional Laplacian $(-\Delta)^s$) arise in problems involving anomalous diffusion (in probability theory, physics, biology and finance).
- Example: A continuous limit of discrete, long jump random walks can be described by the fractional diffusion equation

$$\partial_t u + (-\Delta)^s u = 0.$$

- Motivation: Can we have a fractional analogue of the classical Calderón problem?

Definition of $(-\Delta)^s$ $(0 < s < 1)$:

- (Fourier transform definition)

$$(-\Delta)^s u(x) := \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u(\xi))(x)$$

  where $\mathcal{F}$ denotes the Fourier transform.

- (Singular integral definition)

$$(-\Delta)^s u(x) := c_{n,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dy$$

  where $B_\epsilon(x)$ denotes the open ball centered at x with radius ϵ.

2. Fractional Calderón Problem

- (Caffarelli-Silvestre extension definition)

$$((-\Delta)^s f)(x) := c_s \lim_{y \to 0^+} y^{1-2s} \partial_y u(x,y)$$

where u is the solution of the extension problem

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

- Unique continuation property of $(-\Delta)^s$ has been proven based on CS definition and Carleman estimates (Rüland, 15).

(Ghosh–Salo–Uhlmann, 16)

Let $0 < s < 1$ and $u \in H^s(\mathbb{R}^n)$. Let W be nonempty and open. If

$$(-\Delta)^s u = u = 0 \quad \text{in } W,$$

then $u = 0$ in $\mathbb{R}^n$.

2.1. Fractional Calderón Problem (Schrödinger Type)

Formulation of the fractional Calderón problem:

- We consider the exterior Dirichlet problem

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \qquad u|_{\Omega_e} = g$$

  where $\Omega_e := \mathbb{R}^n \setminus \bar{\Omega}$ and define the Dirichlet-to-Neumann map

$$\Lambda_q : g \to (-\Delta)^s u|_{\Omega_e}.$$

- The knowledge of Λ_q is equivalent to the knowledge of the nonlocal Neumann operator

$$(\mathcal{N}_s u)(x) := c_{n,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy, \quad x \in \Omega_e,$$

  which approaches the classical Neumann derivative as $s \to 1$.

2.1. Fractional Calderón Problem (Schrödinger Type)

Inverse problem: Can we determine q from Λ_q?

Fundamental uniqueness theorem:

(Ghosh–Salo–Uhlmann, 16)

Suppose $n \geq 2$. Let $0 \leq q_{1,2} \in L^\infty(\Omega)$ and let $W_{1,2} \subset \Omega_e$ be nonempty and open. If

$$\Lambda_{q_1} g|_{W_2} = \Lambda_{q_2} g|_{W_2}, \qquad g \in C_c^\infty(W_1),$$

then $q_1 = q_2$ in Ω.

Main ingredients of the proof:

- Integral identity for Dirichlet-to-Neumann maps
- Runge approximation property (based on UCP)

2.2. Fractional Calderón Problem (Conductivity Type)

Assume $0 < \gamma \in C^\infty(\mathbb{R}^n)$ and $\gamma = 1$ in Ω_e. Let

$$L_\gamma := -\mathrm{div}(\gamma(x)\nabla).$$

We define

$$L_\gamma^s := \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-tL_\gamma} - \mathrm{Id}) \frac{dt}{t^{1+s}}$$

where Γ is the Gamma function.

- This semigroup approach works for more general self-adjoint non-negative operators.
- L_γ^s coincides with $(-\Delta)^s$ when $\gamma = 1$ in $\mathbb{R}^n$.

2.2. Fractional Calderón Problem (Conductivity Type)

Formulation of the fractional Calderón problem:

- We consider the exterior Dirichlet problem

$$L_\gamma^s u = 0 \text{ in } \Omega, \qquad u|_{\Omega_e} = g$$

  and define the Dirichlet-to-Neumann map

$$\Lambda_\gamma : g \to L_\gamma^s u|_{\Omega_e}.$$

- Inverse problem: Can we determine γ from Λ_γ?
- Answer: Yes (Ghosh–Uhlmann, 21).

2.2. Fractional Calderón Problem (Conductivity Type)

Main ingredients of the proof:

- Unique continuation results for the classical parabolic equation
- The generalized UCP for the fractional operator:

(Ghosh–Lin–Xiao, 17; Ghosh–Uhlmann, 21)

Let $0 < s < 1$ and $u \in H^s(\mathbb{R}^n)$. Let W be nonempty and open. If

$$L_\gamma^s u = u = 0 \quad \text{in } W,$$

then $u = 0$ in $\mathbb{R}^n$.

- Reduction to the classical Calderón problem

2. Fractional Calderón Problem

Variants of the fractional Calderón problem:

- Local perturbation of fractional Laplacian (Cekić-Lin-Rüland, 18; Covi-Mönkkönen-Railo-Uhlmann, 20)
- Nonlocal perturbation of fractional Laplacian (Bhattacharyya-Ghosh-Uhlmann, 20; Covi, 21)
- Space-time fractional parabolic operator (Lai-Lin-Rüland, 20)
- Fractional magnetic operators (Covi, 19; L, 20; Lai-Zhou, 21)
- Fractional elasticity (L, 21)
- Operators involving fractional gradients (Covi, 18; Lai-Ohm, 20; Railo-Zimmermann, 22)
- Fractional operators on closed manifolds (Feizmohammadi-Ghosh-Krupchyk-Uhlmann, 21; Quan-Uhlmann, 22)

3.1. Nonlinear Fractional Parabolic Problem 1

We consider the power type nonlinear fractional parabolic equation

$$\partial_t u + (-\Delta)^s u + a(x,t,u) = 0$$

where the nonlinearity satisfies

$$a(x,t,z) = \sum_{k=1}^{m} a_k(x,t)|z|^{b_k} z, \tag{1}$$

$0 \leq a_k \in C(\bar{\Omega} \times [0,T])$ and the powers $0 < b_1 < \cdots < b_m$ are not necessarily integers.

3.1. Nonlinear Fractional Parabolic Problem 1

Formulation of the inverse problem:

- We consider the initial exterior problem

$$\partial_t u + (-\Delta)^s u + a(x,t,u) = 0 \quad \text{in } \Omega \times (0,T),$$

$$u = g \quad \text{in } \Omega_e \times (0,T), \qquad u = 0 \quad \text{in } \Omega \times \{0\}.$$

- We define the Dirichlet-to-Neumann map

$$\Lambda_a : g \to (-\Delta)^s u|_{\Omega_e \times (0,T)}.$$

3.1. Nonlinear Fractional Parabolic Problem 1

Inverse problem: Can we determine the nonlinearity a from partial measurements of Λ_a?

The following is the main theorem:

(L, 21)

Let $W_{1,2} \subset \Omega_e$ be nonempty and open. Let $a^{(1,2)}$ be nonlinearities of the form (1). Suppose

$$\Lambda_{a^{(1)}} g|_{W_2 \times (0,T)} = \Lambda_{a^{(2)}} g|_{W_2 \times (0,T)}$$

for small $g \in C_c^\infty(W_1 \times (0, T))$. Then

$$a_k^{(1)} = a_k^{(2)} \text{ in } \Omega \times (0, T), \quad k = 1, \cdots, m.$$

3.1. Nonlinear Fractional Parabolic Problem 1

Forward problem part:

- We make the substitution $w := u - g$ and study the I.V.P.

$$\partial_t w + (-\Delta)^s w + a(x,t,w) = f \text{ in } \Omega \times (0,T), \;\; w = 0 \text{ in } \Omega \times \{0\}.$$

- We consider the semigroup $\{S_\Omega(t)\}_{t\geq 0}$ associated with

$$\partial_t w + (-\Delta)^s w = 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

$$w = 0 \quad \text{in } \Omega_e \times \mathbb{R}_+, \qquad w = w_0 \quad \text{in } \Omega \times \{0\}$$

and for fixed f we define the nonlinear map

$$(Fu)(x,t) := \int_0^t S_\Omega(t-\tau)(f(x,\tau) - a(x,\tau,u(x,\tau)))\,d\tau.$$

3.1. Nonlinear Fractional Parabolic Problem 1

- Based on estimates on $S_\Omega(t)$, we can use fixed-point theorem for F to construct the solution of I.V.P. in the space

$$X := C([0, T]; L^r(\Omega)) \cap L^q(0, T; L^p(\Omega)).$$

for small f. Here p, q, r depend on n, s, a.

- We can prove the L^∞ inequality: Let u be a solution of

$$\partial_t u + (-\Delta)^s u + a(x, t, u) = f \quad \text{in } \Omega \times (0, T),$$

$$u = g \quad \text{in } \Omega_e \times (0, T), \qquad u = 0 \quad \text{in } \Omega \times \{0\}.$$

Then we have

$$||u||_{L^\infty} \leq T||f||_{L^\infty(\Omega\times(0,T))} + ||g||_{L^\infty(\Omega_e\times(0,T))}.$$

3.1. Nonlinear Fractional Parabolic Problem 1

Inverse problem part:

- Unique continuation property of $(-\Delta)^s$
- Runge approximation property associated with $\partial_t + (-\Delta)^s$
- First order linearization (relating the nonlinear problem to the linear one)

Remark:

- The higher order multiple-fold linearization method is widely used in solving inverse problems for power type classical equations (Kurylev-Lassas-Uhlmann, 18; Lassas-Uhlmann-Wang, 18; Krupchyk-Uhlmann, 20; Feizmohammadi-Oksanen, 20; Liimatainen-Lin-Salo-Tyni, 20; Uhlmann-Zhai, 21).

3.1. Nonlinear Fractional Parabolic Problem 1

First order Linearization:

- Let u_g be the solution of

$$\partial_t u + (-\Delta)^s u = 0 \quad \text{in } \Omega \times (0, T),$$

$$u = g \quad \text{in } \Omega_e \times (0, T), \quad u = 0 \quad \text{in } \Omega \times \{0\}.$$

- Let $u_{\lambda,g}$ be the solution of

$$\partial_t u + (-\Delta)^s u + a(x, t, u) = 0 \quad \text{in } \Omega \times (0, T),$$

$$u = \lambda g \quad \text{in } \Omega_e \times (0, T), \quad u = 0 \quad \text{in } \Omega \times \{0\}.$$

- We can show that

$$\frac{u_{\lambda,g}}{\lambda} \to u_g$$

as $\lambda \to 0$ in $L^\infty(\Omega \times (0, T))$.

3.1. Nonlinear Fractional Parabolic Problem 1

Runge approximation property:

Based on UCP of $(-\Delta)^s$, we can prove the parabolic RAP

(Rüland–Salo, 17)

Let $W \subset \Omega_e$ be nonempty and open. Then the set

$$S := \{u_g|_{\Omega\times(0,T)} : g \in C_c^\infty(W \times (0,T))\}$$

is dense in $L^2(\Omega \times (0,T))$.

3.1. Nonlinear Fractional Parabolic Problem 1

Sketch the proof of the main theorem:

- Let $u^{(j)}_{\lambda,g}$ be the solution of

$$\partial_t u + (-\Delta)^s u + a^{(j)}(x,t,u) = 0 \quad \text{in } \Omega \times (0,T),$$

$$u = \lambda g \quad \text{in } \Omega_e \times (0,T), \qquad u = 0 \quad \text{in } \Omega \times \{0\}.$$

- Based on the assumption on DN maps and UCP, we can show

$$u^{(1)}_{\lambda,g} = u^{(2)}_{\lambda,g} =: u_{\lambda,g}$$

in $\mathbb{R}^n \times (0,T)$. Then we have

$$(a_1^{(1)}(x,t) - a_1^{(2)}(x,t))|u_{\lambda,g}|^{b_1} u_{\lambda,g} = R_1^{(2)}(x,t,u_{\lambda,g}) - R_1^{(1)}(x,t,u_{\lambda,g})$$

in $\Omega \times (0,T)$ where

$$R_1^{(i)}(x,t,z) := \sum_{k=2}^{m} a_k^{(i)}(x,t)|z|^{b_k} z.$$

3.1. Nonlinear Fractional Parabolic Problem 1

Sketch the proof of the main theorem (continued):

- We can write $|a_1^{(1)}(x,t) - a_1^{(2)}(x,t)|^{\frac{1}{b_1+1}}$ as the sum of

$$|a_1^{(1)}(x,t) - a_1^{(2)}(x,t)|^{\frac{1}{b_1+1}}(1 - \frac{u_{\lambda,g}}{\lambda})$$

  and

$$\frac{1}{\lambda}|a_1^{(1)}(x,t) - a_1^{(2)}(x,t)|^{\frac{1}{b_1+1}} u_{\lambda,g}.$$

- We can choose g s.t. both the first term (by using RAP and linearization) and the second term (by using L^∞ inequality to estimate $R_1^{(i)}$) can be arbitrarily small as $\lambda \to 0$.
- Iteratively, once we have shown $a_j^{(1)} = a_j^{(2)}$ $(1 \le j \le m' - 1)$, we repeat the process above to show $a_{m'}^{(1)} = a_{m'}^{(2)}$.

3.2. Nonlinear Fractional Parabolic Problem 2

Porous medium equations:

- The classical porous medium equation

$$\partial_t u - \Delta(|u|^{m-1}u) = 0, \qquad m > 1$$

  appears in models for gas flow through porous media, high-energy physics, population dynamics.

- The fractional porous medium equation

$$\partial_t u + (-\Delta)^s(|u|^{m-1}u) = 0, \qquad m > 1, \quad 0 < s < 1$$

  is a combination of fractional diffusion and porous medium nonlinearities, which describes anomalous diffusion through porous media.

3.2. Nonlinear Fractional Parabolic Problem 2

Formulation of the inverse problem:

- We consider the initial exterior problem

$$\partial_t u + L^s_\gamma(|u|^{m-1}u) + \lambda u = 0 \quad \text{in } \Omega \times (0,T),$$

$$u = g \quad \text{in } \Omega_e \times (0,T), \qquad u = 0 \quad \text{in } \Omega \times \{0\}$$

  where the conductivity γ and the absorption coefficient λ are time-independent.
- We define the Dirichlet-to-Neumann map

$$\Lambda_{\gamma,\lambda} : g \to L^s_\gamma(|u|^{m-1}u)|_{\Omega_e \times (0,T)}.$$

3.2. Nonlinear Fractional Parabolic Problem 2

Inverse problem: Can we determine λ and γ in Ω from partial measurements of $\Lambda_{\gamma,\lambda}$?

The following is the main theorem:

(L, 21)

Let $W_{1,2} \subset \Omega_e$ be nonempty and open. Suppose $\lambda^{(1,2)} \in C^\infty(\bar{\Omega})$, $0 < \gamma^{(1,2)} \in C^\infty(\mathbb{R}^n)$ and $\gamma^{(1)} = \gamma^{(2)} = 1$ in Ω_e. Suppose

$$\Lambda_{\gamma^{(1)},\lambda^{(1)}} g|_{W_2 \times (0,T)} = \Lambda_{\gamma^{(2)},\lambda^{(2)}} g|_{W_2 \times (0,T)}$$

for all g s.t. $g^m \in C_c^\infty(W_1)$. Then $\gamma^{(1)} = \gamma^{(2)}$ and $\lambda^{(1)} = \lambda^{(2)}$ in Ω.

3.2. Nonlinear Fractional Parabolic Problem 2

Forward problem part:

- We make the substitution $w := u - g$ and study the I.V.P.

$$\partial_t w + L^s_\gamma(w^m) + \lambda w = f \quad \text{in } \Omega \times (0, T), \quad w = 0 \quad \text{in } \Omega \times \{0\}.$$

- We define the operator A in $H^{-s}(\Omega)$ by

$$Aw := L^s_\gamma(w^m)|_\Omega, \quad D(A) := \{w \in H^{-s}(\Omega) \cap L^1(\Omega) : w^m \in \tilde{H}^s(\Omega)\}$$

  where

$$\tilde{H}^s(\Omega) := \text{the closure of } C^\infty_c(\Omega) \text{ in } H^s(\mathbb{R}^n)$$

  (so $H^{-s}(\Omega) = \tilde{H}^s(\Omega)^*$).

3.2. Nonlinear Fractional Parabolic Problem 2

- It is known (Bonforte-Sire-Vázquez, 15) that A is maximal monotone (i.e. A is monotone and the Ran(Id$+A$)= $H^{-s}(\Omega)$).

- We apply the theory of monotone operators in Hilbert spaces to show the well-posedness result:
  Let $f \in L^2(0, T; H^{-s}(\Omega))$. Then there exists a unique solution

$$w \in C([0, T]; H^{-s}(\Omega)) \cap H^1(0, T; H^{-s}(\Omega))$$

  of I.V.P. Moreover, $w(t) \in D(A)$ for $t \in (0, T)$.

3.2. Nonlinear Fractional Parabolic Problem 2

Inverse problem part:

- We use a time-integral transform, which relates the nonlinear parabolic problem to the linear elliptic problem. This enables us to apply the uniqueness result for the fractional Calderón problem to determine γ.
- Once γ is determined, we use UCP of L^s_γ to determine λ.

Remark:

- Time-integral transform methods have been used in solving the inverse problem for the classical porous medium equation (Cârstea-Ghosh-Nakamura, 21; Cârstea-Ghosh-Uhlmann, 21).

3.2. Nonlinear Fractional Parabolic Problem 2

Sketch the proof of the main theorem:

- We make substitutions $v := u^m, \tilde{g} := g^m$ to write IEP as

$$\partial_t(v^{\frac{1}{m}}) + L^s_\gamma v + \lambda v^{\frac{1}{m}} = 0 \quad \text{in } \Omega \times (0, T),$$

$$v = \tilde{g} \quad \text{in } \Omega_e \times (0, T), \quad v = 0 \quad \text{in } \Omega \times \{0\}.$$

- We define the associated DN map

$$\tilde{\Lambda}_{\gamma,\lambda} : \tilde{g} \to L^s_\gamma v|_{\Omega_e \times (0,T)}.$$

  The knowledge of $\tilde{\Lambda}_{\gamma,\lambda}$ is equivalent to the knowledge of $\Lambda_{\gamma,\lambda}$.
- $v^{(h)}$ denotes the solution corresponding to $\tilde{g} := hg_0$ for fixed time-independent $g_0 \in C^\infty_c(W_1)$. Here $h > 0$ is a parameter.

Sketch the proof of the main theorem (continued):

- We consider the time-integral transform

$$V(x) := \int_0^T (T-t)^\alpha v(x,t)\,dt.$$

We also define

$$M(x) := \alpha \int_0^T (T-t)^{\alpha-1} v^{\frac{1}{m}}(x,t)\,dt,$$

$$N(x) := \lambda(x) \int_0^T (T-t)^{\alpha} v^{\frac{1}{m}}(x,t)\,dt.$$

- Applying the transform and integrating by parts w.r.t. t, we get

$$L^s_\gamma V^{(h)} = M^{(h)} + N^{(h)} \text{ in } \Omega, \qquad V^{(h)}|_{\Omega_e} = C_\alpha T^{1+\alpha} h g_0.$$

3.2. Nonlinear Fractional Parabolic Problem 2

Sketch the proof of the main theorem (continued):

- We write

$$V^{(h)} = C_\alpha T^{1+\alpha} h V_0 + R^{(h)}$$

  where V_0 is the solution of

$$L^s_\gamma V_0 = 0 \text{ in } \Omega, \qquad V_0|_{\Omega_e} = g_0$$

  and $R^{(h)}$ is the solution of

$$L^s_\gamma R^{(h)} = M^{(h)} + N^{(h)} \text{ in } \Omega, \qquad R^{(h)}|_{\Omega_e} = 0.$$

- We can estimate $M^{(h)}, N^{(h)}$ and then show that

$$R^{(h)} = O(h^{\frac{1}{m}})$$

  as $h \to \infty$ in H^s-norm.

3.2. Nonlinear Fractional Parabolic Problem 2

Sketch the proof of the main theorem (continued):

- Applying L^s_γ to the equality, we get

$$h^{-1} L^s_\gamma V^{(h)} = C_\alpha T^{1+\alpha} L^s_\gamma V_0 + h^{-1} L^s_\gamma R^{(h)}.$$

- Let $h \to \infty$. Then we see that $\Lambda^{lin}_\gamma g_0|_{W_2}$ is determined by the time-integral of $\tilde{\Lambda}_{\gamma,\lambda}\tilde{g}|_{W_2\times(0,T)}$.

- Now we apply the uniqueness theorem for the fractional Calderón problem to determine γ.

Sketch the proof of the main theorem (continued):

- Pick a nonzero g. Let u_j ($j = 1, 2$) be the solution of

$$\partial_t u + L^s_\gamma(u^m) + \lambda_j u = 0 \quad \text{in } \Omega \times (0, T),$$

$$u = g \quad \text{in } \Omega_e \times (0, T), \qquad u = 0 \quad \text{in } \Omega \times \{0\}.$$

- Using UCP, we can show

$$u_1 = u_2 := u \ \text{ in } \mathbb{R}^n \times (0, T)$$

and for any $x_0 \in \Omega$, we can choose $(x_k, t_k) \in \Omega \times (0, T)$ s.t. $x_k \to x_0$ and $u(x_k, t_k) \neq 0$. Hence

$$\lambda^{(j)}(x_0) = \lim_{k\to\infty} \lambda^{(j)}(x_k)$$

$$= -\lim_{k\to\infty} \frac{\partial_t u(x_k, t_k) + L^s_\gamma(u^m)|_{(x_k,t_k)}}{u(x_k, t_k)}, \qquad j = 1, 2.$$

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