Inverse fractional conductivity problem

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UCI IP Seminar, Sep 2022

Outline

1. Inverse (fractional) conductivity problem
2. Nonlocal Calderón problems
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4. Counterexamples for disjoint sets of measurements
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Collaborators and acknowledgements

- The talk is mainly based on collaboration with **Philipp Zimmermann** (PhD candidate, ETH Zürich).
- Some parts are joint works **Giovanni Covi** (Heidelberg), **Manas Kar** (IISER Bhopal) and **Teemu Tyni** (Toronto).
- The author is supported by the **Vilho, Yrjö and Kalle Väisälä Foundation of the Finnish Academy of Science and Letters**.

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Inverse conductivity problem (Calderón, 1980)

- Is it possible to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary?

$$\nabla \cdot (\gamma \nabla u)|_{\Omega} = 0, \quad u|_{\partial\Omega} = f.$$

- Suppose one knows the DN map $\Lambda_\gamma f = \gamma \partial_\nu u|_{\partial\Omega}$, can we determine the electrical conductivity $\gamma : \Omega \to \mathbb{R}$ uniquely?
- Mathematical model for the **electrical impedance tomography** (EIT).

Classical Calderón problem ($n \geq 3$)

- Boundary determination ($\Rightarrow$ uniqueness for real-analytic γ) (Kohn–Vogelius, 1984).
- Interior uniqueness when $n \geq 3$ (Sylvester–Uhlmann, 1987).
- A reconstruction method (Nachman, 1988).
- Logarithmic stability (Alessandrini, 1988) and optimality (Mandache, 2001).
- Studied typically via the **Liouville transformation**:

$$-\nabla \cdot \gamma \nabla(\gamma^{-1/2} u) = \gamma^{1/2}(-\Delta + q)u, \quad q = \gamma^{-1/2}\Delta(\gamma^{1/2}).$$

- The inverse problem is then solved using the **complex geometric optics** (CGO) solutions and their behaviour when $|\zeta| \to \infty$:

$$u(x) = e^{i\zeta \cdot x}(1 + r_\zeta(x)).$$

Some basic definitions

- We say that an open set $\Omega_\infty \subset \mathbb{R}^n$ of the form $\Omega_\infty = \mathbb{R}^{n-k} \times \omega$, where $n \geq k > 0$ and $\omega \subset \mathbb{R}^k$ is a bounded open set, is a **cylindrical domain**.
- We say that an open set $\Omega \subset \mathbb{R}^n$ is **bounded in one direction** if there exists a cylindrical domain $\Omega_\infty \subset \mathbb{R}^n$ and a rigid Euclidean motion $A(x) = Lx + x_0$, where L is a linear isometry and $x_0 \in \mathbb{R}^n$, such that $\Omega \subset A\Omega_\infty$.
- The **fractional gradient** is defined for all sufficiently regular functions by the formula

$$\nabla^s u(x,y) = \sqrt{\frac{C_{n,s}}{2}} \frac{u(x) - u(y)}{|x-y|^{n/2+s+1}}(x-y)$$

and div_s denotes its adjoint operator. In particular, $\mathrm{div}_s(\nabla^s u) = (-\Delta)^s u$ in the weak sense for all $u \in H^s(\mathbb{R}^n)$.

A fractional Poincaré inequality

Theorem (Poincaré inequality, R.–Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be an open set that is bounded in one direction. Suppose that $2 \leq p < \infty$ and $0 \leq s \leq t < \infty$, or $1 < p < 2$, $1 \leq t < \infty$ and $0 \leq s \leq t$. Then there exists $C(n,p,s,t,\Omega) > 0$ such that

$$\|(-\Delta)^{s/2}u\|_{L^p(\mathbb{R}^n)} \leq C\|(-\Delta)^{t/2}u\|_{L^p(\mathbb{R}^n)}$$

for all $u \in \tilde{H}^{t,p}(\Omega)$.

Conjecture (Equivalence of the optimal constants)

Let $\Omega \subset \mathbb{R}^n$ be an open (bounded) domain and $1 < p < \infty$. If $C_{r,z}$ is the optimal fractional Poincaré constant for $r > z \geq 0$, then $C_{t,s} = C_{r,z}^{\frac{t-s}{r-z}}$ is the optimal Poincaré constant for $t > s \geq 0$.

Fractional conductivity equation

- Let $s \in (0,1)$ and consider the Dirichlet problem for the **fractional conductivity equation**:

$$\begin{aligned} \mathrm{div}_s(\Theta_\gamma \nabla^s u) &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{in } \Omega_e, \end{aligned} \tag{1}$$

  where $\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$ is the exterior of the domain Ω, Θ_γ is an appropriate matrix depending on the **global, elliptic, conductivity** $\gamma \in L^\infty_+(\mathbb{R}^n)$.

- We say $u \in H^s(\mathbb{R}^n)$ is a (weak) **solution** of (1) if the bilinear form

$$B_\gamma(u,\phi) := \frac{C_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{\gamma^{1/2}(x)\gamma^{1/2}(y)}{|x-y|^{n+2s}} (u(x)-u(y))(\phi(x)-\phi(y))\, dxdy$$

  vanishes for all $\phi \in C_c^\infty(\Omega)$ and $u - f \in \widetilde{H}^s(\Omega) := \overline{C_c^\infty(\Omega)}^{H^s(\mathbb{R}^n)}$.

Inverse fractional conductivity problem

- Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\gamma \in L^\infty(\mathbb{R}^n)$ satisfy $\gamma \geq \gamma_0 > 0$.
- For all $f \in X := H^s(\mathbb{R}^n)/\widetilde{H}^s(\Omega)$ there are unique weak solutions $u_f \in H^s(\mathbb{R}^n)$ of the fractional conductivity equation

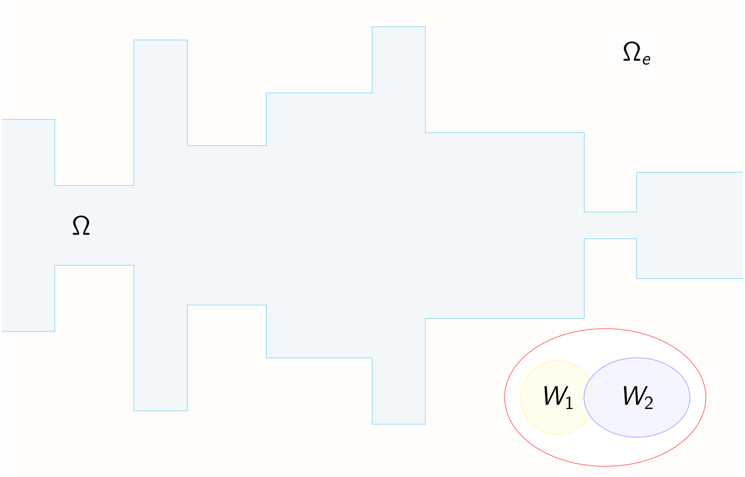
$$\begin{aligned} \mathrm{div}_s(\Theta \nabla^s u) &= 0 \quad \text{in} \quad \Omega, \\ u &= f \quad \text{in} \quad \Omega_e. \end{aligned}$$

- The **exterior DN maps** $\Lambda_\gamma \colon X \to X^*$ given by

$$\langle \Lambda_\gamma f, g \rangle := B_\gamma(u_f, g),$$

  where $u_f \in H^s(\mathbb{R}^n)$ is the unique solution to the fractional conductivity equation, is a well-defined bounded linear map.
- The **inverse fractional conductivity problem** asks: Suppose that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, does it imply that $\gamma_1 = \gamma_2$?

Geometric illustration of related domains



Comments on the approximation properties

- As $s \uparrow 1$, then the fractional conductivity operator converges in the sense of distributions to the classical conductivity operator when applied to sufficiently regular functions (Covi, 2021).
- *Approximation properties with respect to the DN maps and inverse problems require more work and understanding.*
- There is a work by Ghosh–Uhlmann (2021) showing that if the exterior Cauchy data of fractional powers of elliptic 2nd order operators agree for $0 < s < 1$, then also the boundary Cauchy data agree. (Their and our settings are however different.) *Could there be any hope for reversing this fascinating connection?*

Terminology for abstract nonlocal Calderón's problems

Let $s \in \mathbb{R}$ and $B\colon H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$ be a bounded bilinear form:

(i) We say that B has the **left UCP** on an open nonempty set $W \subset \mathbb{R}^n$ when the following holds: If $u \in H^s(\mathbb{R}^n)$, $u|_W = 0$ and $B(u, \phi) = 0$ for all $\phi \in C_c^\infty(W)$, then $u \equiv 0$.

(ii) We say that B has the **right UCP** on an open nonempty set $W \subset \mathbb{R}^n$ when the following holds: If $u \in H^s(\mathbb{R}^n)$, $u|_W = 0$ and $B(\phi, u) = 0$ for all $\phi \in C_c^\infty(W)$, then $u \equiv 0$.

(iii) We say that B is **local** when the following holds: If $u, v \in H^s(\mathbb{R}^n)$ and $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$, then $B(u, v) = 0$.

Abstract nonlocal Calderón problems

Lemma

Let $s \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^n$ be open set such that $\Omega_e \neq \emptyset$. Let $B\colon H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$ be a bounded bilinear form that is (strongly) coercive in $\tilde{H}^s(\Omega)$, that is, there exists some $c > 0$ such that $B(u,u) \geq c\|u\|^2_{H^s(\mathbb{R}^n)}$ for all $u \in \tilde{H}^s(\Omega)$. Then the following hold:

1. *Existence of solutions: For any $f \in H^s(\mathbb{R}^n)$ and $F \in (\tilde{H}^s(\Omega))^*$ there exists a unique $u \in H^s(\mathbb{R}^n)$ such that $u - f \in \tilde{H}^s(\Omega)$ and $B(u,\phi) = F(\phi)$ for all $\phi \in \tilde{H}^s(\Omega)$. When $F \equiv 0$, we denote this unique solution by u_f.*
2. *Let $X := H^s(\mathbb{R}^n)/\tilde{H}^s(\Omega)$ be the abstract trace space. Then the exterior DN map $\Lambda_B\colon X \to X^*$ defined by $\Lambda_B[f][g] := B(u_f, g)$ for $[f],[g] \in X$ is a well-defined bounded linear map.*

Runge approximation property

One may prove the following functional analytic theorem using the ideas of Ghosh–Salo–Uhlmann (2020), Cekić–Lin–Rüland (2020), Covi–Mönkkönen–R.–Uhlmann (2022):

Theorem (R.-Zimmermann, 2022)

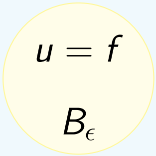
Let $s \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ be an open set such that $\Omega_e \neq \emptyset$. Let $L, q\colon H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$ be bounded bilinear forms and assume that q is local and that $B_{L,q} := L + q$ is (strongly) coercive in $\tilde{H}^s(\Omega)$.

- (i) *If L has the right UCP on a nonempty open set $W \subset \Omega_e$, then $\mathcal{R}(W) := \{\, u_f - f \,;\, f \in C_c^\infty(W) \,\} \subset \tilde{H}^s(\Omega)$ is dense.*
- (ii) *If L has the left UCP on a nonempty open set $W \subset \Omega_e$, then $\mathcal{R}^*(W) := \{\, u_g^* - g \,;\, g \in C_c^\infty(W) \,\} \subset \tilde{H}^s(\Omega)$ is dense.*

Example (R.–Zimmermann, 2022)

Let us denote $B_\epsilon = B(0;\epsilon) \subset \mathbb{R}^n$ for any $\epsilon > 0$ and $n \geq 1$. For any $\epsilon, \delta > 0$, $s \in \mathbb{R}_+ \setminus \mathbb{Z}$ and $\Omega := \mathbb{R}^n \setminus \overline{B_\epsilon}$, the restriction to $\mathbb{R}^n \setminus \overline{B_\epsilon}$ of the unique solutions u_f to the equation $((-\Delta)^s + \delta)u = 0$ in $\mathbb{R}^n \setminus \overline{B_\epsilon}$ are dense in $\tilde{H}^s(\mathbb{R}^n \setminus \overline{B_\epsilon})$ with exterior conditions $f \in C_c^\infty(B_\epsilon)$.

$$((-\Delta)^s + \delta)u = 0$$



Generalized Ghosh–Salo–Uhlmann theorem

Theorem (R.-Zimmermann, 2022)

Let $s \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^n$ be open such that $\Omega_e \neq \emptyset$. Let $L\colon H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$ be a bounded bilinear form with the following properties:

1. *There exists a nonempty open set $W_1 \subset \Omega_e$ such that L has the right UCP on W_1.*
2. *There exists a nonempty open set $W_2 \subset \Omega_e$ such that L has the left UCP on W_2.*
3. *$W_1 \cap W_2 = \emptyset$.*

Let $q_j\colon H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$, $j = 1, 2$, be local and bounded bilinear forms. Suppose that $B_{L,q_j} = L + q_j$ are (strongly) coercive in $\tilde{H}^s(\Omega)$. If the exterior data $\Lambda_{L,q_1}[f][g] = \Lambda_{L,q_2}[f][g]$ agree for all $f \in C_c^\infty(W_1)$ and $g \in C_c^\infty(W_2)$, then $q_1 = q_2$ in $\tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega)$.

Examples from the literature

- $(-\Delta)^s + w$ where $w \in L^\infty(\Omega)$ and Ω is bounded where $L(u,v) = ((-\Delta)^{s/2}u, (-\Delta)^{s/2}v)$ and $q(u,v) = \int_{\mathbb{R}^n} wuvdx$ (Ghosh–Salo–Uhlmann, 2016). An extension to certain Sobolev multiplier perturbations w (Rüland–Salo, 2017).
- $L^s + w$ where L^s is a fractional power of an elliptic 2nd order operator L and $w \in L^\infty(\Omega)$ and Ω is bounded (Ghosh–Lin–Xiao, 2017).
- $(-\Delta)^s + w + c \cdot \nabla$, c a vector field, has 0th and 1st order terms (Cekić–Lin–Rüland, 2018).
- Extension for general local linear lower order perturbations $(-\Delta)^s + P$, $s \in \mathbb{R}_+ \setminus \mathbb{Z}$, $m \in \mathbb{N}$ such that $2s > m$, by $P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ in $\alpha_\alpha \in M_0(H^{s-|\alpha|} \to H^{-s})$ (Covi–Mönkkönen–R.–Uhlmann, 2021).
- ...and much more

New examples (R.–Zimmermann, 2022)

- (Domains without Poincaré inequalities) For $(-\Delta)^s + q$ in Ω where $s \in \mathbb{R}_+ \setminus \mathbb{Z}$ and the potential q is uniformly positive and bounded, i.e. $q \in L^\infty_+(\mathbb{R}^n)$.
- (Higher order perturbations) For $(-\Delta)^t + (-\Delta)^{s/2}(\gamma(-\Delta)^{s/2}\cdot) + q$ in Ω where $t \in \mathbb{R}_+ \setminus \mathbb{Z}$, $s \in 2\mathbb{Z}$ and $t < s$, and $\gamma, q \in L^\infty_+(\mathbb{R}^n)$.
- (A small fractional perturbation of the conductivity equation – with exterior data) $\lambda(-\Delta)^t + \mathrm{div}(\gamma\nabla\cdot)$ where $\lambda, t \in (0,1)$, $\gamma \in L^\infty_+(\mathbb{R}^n)$, Ω bounded in one direction. (One can plug in an elliptic $L^\infty(\Omega; \mathbb{R}^{n\times n})$ anisotropic conductivity as well.)
- Solutions to the related exterior value problems are dense in the corresponding spaces $\tilde{H}^s(\Omega)$, $\tilde{H}^s(\Omega)$ and $\tilde{H}^1(\Omega)$, respectively.
- ...many other results extend to domains bounded in one direction.

Solving the inverse fractional conductivity problem

Define $m_\gamma := \gamma^{1/2} - 1$ and call it the *background deviation* of γ.

Theorem (R.–Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ are uniformly elliptic with $m_1, m_2 \in H^{s,n/s}(\mathbb{R}^n)$. Suppose that $W \subset \Omega_e$ is a nonempty open set such that γ_1, γ_2 are continuous a.e. in W. Then $\Lambda_{\gamma_1} f|_W = \Lambda_{\gamma_2} f|_W$ for all $f \in C_c^\infty(W)$ if and only if $\gamma_1 = \gamma_2$ in $\mathbb{R}^n$.

- When $m \in H^{2s,n/2s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ earlier by Covi–R.–Zimmermann (2022).
- Brown conjectured (2003) that the classical Calderón problem is solvable for $W^{1,p}(\Omega)$ conductivities when $p > n$ and Haberman proved (2014) uniqueness when $\gamma \in W^{1,n}(\Omega)$, $n = 3, 4$.

Two fundamental properties of DN maps

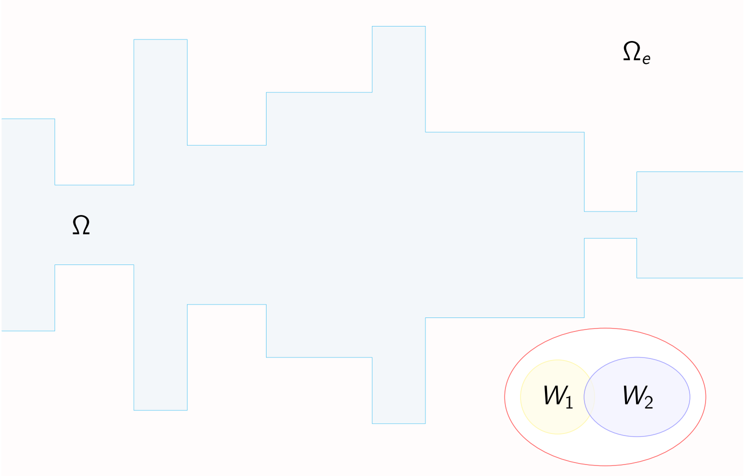
Theorem (Covi–R.–Zimmermann, R.–Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ are uniformly elliptic with $m_1, m_2 \in H^{s,n/s}(\mathbb{R}^n)$. Assume that $W_1, W_2 \subset \Omega_e$ are nonempty open sets and that $\gamma_1|_{W_1 \cup W_2} = \gamma_2|_{W_1 \cup W_2}$ holds. If $W_1 \cap W_2 \neq \emptyset$, then $\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$ if and only if $\gamma_1 = \gamma_2$ in $\mathbb{R}^n$.

Theorem (Covi–R.–Zimmermann, R.–Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < 1$. Assume that $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ satisfy $\gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0$. Suppose that $W \subset \Omega_e$ is a nonempty open set such that γ_1, γ_2 are continuous a.e. in W. If $\Lambda_{\gamma_1} f|_W = \Lambda_{\gamma_2} f|_W$ for all $f \in C_c^\infty(W)$, then $\gamma_1 = \gamma_2$ a.e. in W.

Recall the picture:



UCP of the DN maps 1/2

(i) **Low regularity fractional Liouville reduction** when $\gamma \in L^\infty_+(\mathbb{R}^n)$, $m \in H^{s,n/s}(\mathbb{R}^n)$:

$$\langle \Theta_\gamma \nabla^s u, \nabla^s \phi \rangle_{L^2(\mathbb{R}^{2n})} = \langle (-\Delta)^{s/2}(\gamma^{1/2}u), (-\Delta)^{s/2}(\gamma^{1/2}\phi) \rangle_{L^2(\mathbb{R}^n)} + \langle q_\gamma(\gamma^{1/2}u), (\gamma^{1/2}\phi) \rangle, \quad u, \phi \in H^s(\mathbb{R}^n)$$

where

$$\langle q_\gamma u, \phi \rangle = -\langle (-\Delta)^{s/2} m, (-\Delta)^{s/2}(\gamma^{-1/2} u\phi) \rangle_{L^2(\mathbb{R}^n)}$$

is a suitable Sobolev multiplier in $M(H^s \to H^{-s})$.

(ii) **Reduction of DN maps**: If $\gamma_1|_{W_1 \cup W_2} = \gamma_2|_{W_1 \cup W_2}$ and $\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$, then $\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2}$.

(iii) **Fractional Calderón problem for globally defined singular potentials** (Ghosh–Salo–Uhlmann, Rüland–Salo): If $\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$, then $q_1 = q_2$ in Ω.

UCP of the DN maps 2/2

(i) **Exterior determination** for the **fractional Schrödinger equation**: $\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$ and $W = W_1 \cap W_2 \neq \emptyset$, then $q_1 = q_2$ in W. This uses the earlier interior determination step, which already guarantees that $q_1 = q_2$ in Ω.

(ii) We may then use the assumption that $\gamma_1|_W = \gamma_2|_W$ and the knowledge (in the sense of distributions/as Sobolev multipliers)

$$-\frac{(-\Delta)^s(\gamma_1^{1/2}-1)}{\gamma_1^{1/2}} = q_1 = q_2 = -\frac{(-\Delta)^s(\gamma_2^{1/2}-1)}{\gamma_2^{1/2}} \quad \text{in } W$$

and a **UCP of the fractional Laplacians**: If $u \in H^{r,p}(\mathbb{R}^n)$ for $r \in \mathbb{R}, p \in [1,\infty)$ and $(-\Delta)^t u = u = 0$ in a nonempty open $V \subset \mathbb{R}^n$, $t \in \mathbb{R}_+ \setminus \mathbb{N}$, then $u \equiv 0$ in $\mathbb{R}^n$ (Kar–R.–Zimmermann, 2022 + based on several other works). Here $p = n/s > 2$.

(iii) Altogether, $\gamma_1 \equiv \gamma_2$.

Exterior determination 1/2

(i) Define the **Dirichlet energy** first as

$$E_\gamma(u) := B_\gamma(u,u) = \int_{\mathbb{R}^{2n}} \Theta_\gamma \nabla^s u \cdot \nabla^s u \, dxdy.$$

Notice that $E_\gamma(u_f) = \langle \Lambda_\gamma f, f \rangle_{X^* \times X}$ where u_f is the unique solution of the fractional conductivity equation with the exterior condition f.

(ii) **Elliptic estimate**: Let $W \subset \Omega_e$, $\mathrm{dist}(W,\Omega) > 0$, $|W| < \infty$. If $f \in C_c^\infty(W)$ and $u_f \in H^s(\mathbb{R}^n)$ is the unique solution of

$$\begin{aligned} ((-\Delta)^s + q)u &= 0 \quad \text{in} \quad \Omega, \\ u &= f \quad \text{in} \quad \Omega_e, \end{aligned}$$

then

$$\|u_f|_\Omega\|_{\tilde{H}^s(\Omega)} = \|u_f - f\|_{H^s(\mathbb{R}^n)} \leq C\|f\|_{L^2(W)}$$

for some $C(n, s, |W|, \Omega, \mathrm{dist}(W,\Omega)) > 0$.

Exterior determination 2/2

(i) This uses the quadratic definition of the fractional Laplacian

$$\langle (-\Delta)^s f, \phi \rangle = \frac{C_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{(f(x)-f(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}} dxdy.$$

Similar argument can be made for the conductivity equation.

(ii) **Construction of special solutions:** $\phi_N \in C_c^\infty(W)$ such that $\|\phi_N\|_{L^2(W)} \to 0$ as $N \to \infty$ and $\|\phi_N\|_{H^s(\mathbb{R}^n)} = 1$ for all $N \in \mathbb{N}$. Let $u_N \in H^s(\mathbb{R}^n)$ be the unique solutions to the conductivity equation with $u_N|_{\Omega_e} = \phi_N$. The elliptic energy estimate and the given properties of the exterior conditions give that $E_\gamma(u_N)$ and $E_\gamma(\phi_N)$ are equal as $N \to \infty$. These exterior conditions are similar to the boundary conditions considered by Kohn and Vogelius (1984).

(iii) **Energy concentration property**: Given any $x_0 \in W$, one may show that there exists such sequences ϕ_N so that $E_\gamma(\phi_N) \to \gamma(x_0)$ as $N \to \infty$.

Counterexamples

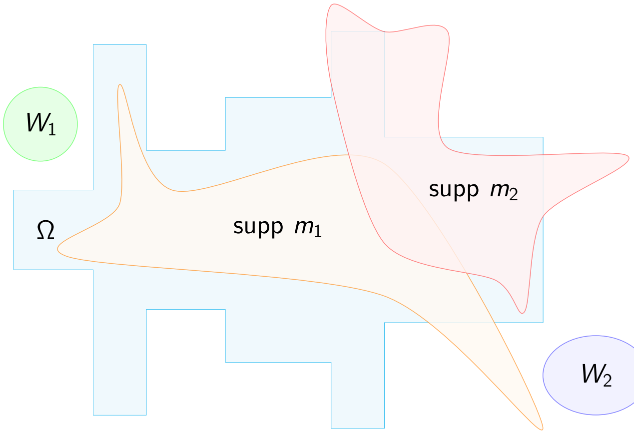
Our uniqueness result for the partial data problem is complemented with the following general counterexamples:

Theorem (R.–Zimmermann, 2022)

*Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction, $0 < s < \min(1, n/2)$. For **any** nonempty open **disjoint sets** $W_1, W_2 \subset \Omega_e$ with $\mathrm{dist}(W_1 \cup W_2, \Omega) > 0$ there exist two different conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ such that $\gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0$, $m_1, m_2 \in H^{s,n/s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$, and $\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$.*

The problem remains open for any nonempty open disjoint sets $W_1, W_2 \subset \Omega_e$ with $\mathrm{dist}(W_1 \cup W_2, \Omega) = 0$.

Graphical illustration



Sketch of the proof 1/2

Using the fractional Liouville reduction one can **characterize** the **invariance of data**, for **any** disjoint data the following holds:

Lemma (R.–Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ with background deviations m_1, m_2 satisfy $\gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0$ and $m_1, m_2 \in H^{s,n/s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$. Finally, assume that $W_1, W_2 \subset \Omega_e$ are nonempty disjoint open sets and that $\gamma_1|_{W_1 \cup W_2} = \gamma_2|_{W_1 \cup W_2}$ holds. Then there holds $\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$ for all $f \in C_c^\infty(W_1)$ ***if and only if*** *$m_0 := m_1 - m_2 \in H^s(\mathbb{R}^n)$ is the unique solution of*

$$\begin{aligned} (-\Delta)^s m + q_{\gamma_2} m &= 0 \quad \text{in} \quad \Omega, \\ m &= m_0 \quad \text{in} \quad \Omega_e. \end{aligned}$$

Sketch of the proof 2/2

- Take $\gamma_2 \equiv 1$. Now, by the invariance of data and searching for $\gamma_1 = (m_1 + 1)^2$, the problem reduces to finding a s-**harmonic function** in Ω, i.e. $m_1 \in H^{s,n/s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ which solves

$$(-\Delta)^s m_1 = 0 \quad \text{in} \quad \Omega, \qquad m_1 = m_0 \quad \text{in} \quad \Omega_e, \tag{2}$$

  with the "positivity" condition $m_1 \geq \gamma_0^{1/2} - 1$ and any suitable exterior condition $m_0 \in C_c^\infty(\Omega_e \setminus \overline{W_1 \cup W_2})$.
- One may first look for a $H^s(\mathbb{R}^n)$ function which is s-harmonic in a slightly larger domain Ω' and vanishes near $\overline{W_1 \cup W_2}$. Using a **mollification argument** one finds a smooth s-harmonic function solving (2) with the right regularity properties, as $n/s > 2$.
- Finally, using the linearity of the equation and a **scaling argument**, the positivity condition can be made to hold.

Sets in the proof

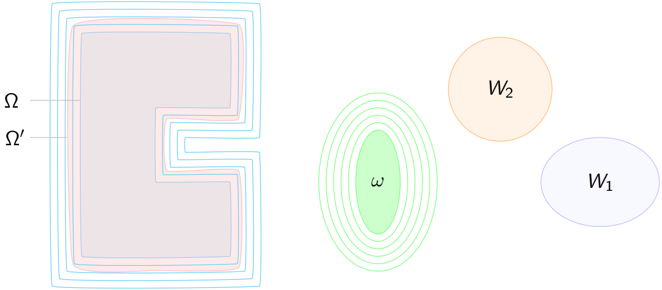


Figure: We construct in the first step a nonzero s-harmonic background deviation $\tilde{m}_1 \in H^s(\mathbb{R}^n)$ in the set Ω', which has a smooth boundary and lies in the deformed annulus $\Omega_{3\epsilon} \setminus \overline{\Omega}_{2\epsilon}$, and then obtain by mollification a nonzero smooth $s-$harmonic function $m_1 := \tilde{m}_1 * \rho_\epsilon$ in the set Ω. The set $\omega \Subset \Omega_e \setminus \overline{W_1 \cup W_2}$ is used to construct a cut–off function $\eta \in C_c^\infty(\omega_{3\epsilon})$ with $\eta|_{\overline{\omega}} = 1$, which $\tilde{m}_1$ has as an exterior value and its support contained in $\Omega_{5\epsilon} \cup \omega_{5\epsilon}$. Next scale so that $\|cm_1\|_{L^\infty(\mathbb{R}^n)} \leq 1/2$ and set $\gamma_0 = 1/4$.

Stability estimate in the exterior

Write $\|A\|_* := \|A\|_{H^s(\Omega_e)\to(H^s(\Omega_e))^*}$. The exterior determination argument is constructive and leads to the following stability estimate:

Theorem (Covi–R.–Zimmermann, R.–Zimmermann, 2022)

Let $\Omega \subset \mathbb{R}^n$ be a domain bounded in one direction and $0 < s < 1$. Assume that $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ satisfy $\gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0$, and are continuous a.e. in Ω_e. There exists a constant $C > 0$ depending only on s such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega_e)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*.$$

The argument is "local" in the exterior. Therefore, similar holds with the partial data in $W \subset \Omega_e$.

Stability estimate in the interior

Theorem (Covi–R.–Tyni–Zimmermann, 2022)

Let $0 < s < \min(1, n/2)$, $\epsilon > 0$ and assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Suppose that the the conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ with background deviations m_1, m_2 fulfill the following conditions:

(i) $\gamma_0 \leq \gamma_1(x), \gamma_2(x) \leq \gamma_0^{-1}$ *for some* $0 < \gamma_0 < 1$,

(ii) $m_1 - m_2 \in H^s(\mathbb{R}^n)$ *and there exist* $C_1, C_2 > 0$ *such that*

$$\|m_i\|_{H^{4s+2\epsilon, \frac{n}{2s}}(\mathbb{R}^n)} \leq C_1, \quad \|(-\Delta)^s m_i\|_{L^1(\Omega_e)} \leq C_2$$

for $i = 1, 2$.

If $\theta_0 \in (\max(1/2, 2s/n), 1)$ *and there holds* $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* \leq 3^{-1/\delta}$ *for some* $0 < \delta < \frac{1-\theta_0}{2}$, *then we have*

$$\|\gamma_1^{1/2} - \gamma_2^{1/2}\|_{L^q(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*)$$

for all $1 \leq q \leq \frac{2n}{n-2s}$, *where* $\omega(x)$ *is a logarithmic modulus of continuity satisfying*

$$\omega(x) \leq C|\log x|^{-\sigma}, \quad \text{for} \quad 0 < x \leq 1,$$

for some constants $\sigma, C > 0$ *depending only on* $s, \epsilon, n, \Omega, C_1, C_2, \theta_0$ *and* γ_0.

About the proof

1. The proof is based on one of the possible uniqueness proofs with full data.
2. The proof uses the **stability estimate for the corresponding Schrödinger problem** by Rüland–Salo (2020).
3. The proof uses the earlier **exterior stability estimate**, which also is related to having $L^1 \subset (L^\infty)^*$ a priori bound in the exterior.
4. Other **key properties** to show (resembling Alessandrini's work) are "$\|\Lambda_{q_1} - \Lambda_{q_2}\|_* \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*$" up to a constant depending of the a priori bounds (the real estimate looks a bit different), and the **identity**

$$\mathrm{div}_s(\Theta_{\gamma_1}\nabla^s \widetilde{m}) = \gamma_1^{1/2}\gamma_2^{1/2}(q_2 - q_1) \quad \text{in} \quad \mathbb{R}^n,$$

where $\widetilde{m} := (\gamma_1^{1/2} - \gamma_2^{1/2})/\gamma_1^{1/2}$.

Open problems 1/2

1. *Regularity in the exterior.* Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ are uniformly elliptic and $W \subset \Omega_e$ is an open set. Does $\Lambda_{\gamma_1} f|_W = \Lambda_{\gamma_2} f|_W$ for all $f \in C_c^\infty(W)$ imply that $\gamma_1 = \gamma_2$ a.e. in W?
2. *Fractional Astala–Päivärinta type theory.* Does the partial/full data uniqueness hold for the uniformly elliptic conductivities that only satisfy $\gamma \in L^\infty(\mathbb{R}^n)$? Can one remove the assumption that conductivities converge to the trivial conductivity at infinity by some other regularity assumptions?
3. *Missing counterexamples.* Are there counterexamples to uniqueness in the partial data inverse problem for all nonempty open sets $W_1, W_2 \subset \mathbb{R}^n$ such that $W_1 \cap W_2 = \emptyset$ and $\mathrm{dist}(W_1 \cup W_2, \Omega) = 0$?

Open problems 2/2

1. *Partial data stability.* Does the partial data stability hold?
2. *Regularity of the boundary and domain.* Do the stability results hold without smoothness or boundedness assumptions?
3. *General kernels.* Under what conditions, on the symbols $a(x,y)$, one may obtain global uniqueness results in $\mathbb{R}^n$ and how to characterize "gauge" for more general equations related to

$$B_a(u,\phi) = \int_{\mathbb{R}^{2n}} \frac{a(x,y)}{|x-y|^{n+2s}}(u(x)-u(y))(\phi(x)-\phi(y))\,dxdy?$$

Our work has extensively analyzed symbols of the product type $a(x,y) = \sigma(x)\sigma(y)$, and the works of Ghosh–Uhlmann and their collaborators some aspects for another classes of kernels generated by the heat semigroups of elliptic operators.

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Thank you for your attention!