# Travel time inverse problems on simple Riemannian manifolds

Boya Liu
(Joint with Joonas Ilmavirta and Teemu Saksala)

North Carolina State University

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# This talk is based on the following manuscript :

*Three travel time inverse problems on simple Riemannian manifolds*, preprint, ArXiv : 2208.08422

# Outline

- Introduction : Simple Manifolds
- Problem 1 : Uniqueness of Broken Scattering Relations
- Problem 2 : Travel Time Data
- Reduction from Broken Scattering Relation to Travel Time Data
- Problem 3 : Travel Time Difference Data

# Simple manifolds

**Conjecture (Michel, 1981)**

Simple Riemannian manifolds are boundary rigid.

A compact Riemannian manifold $(M, g)$ is simple if

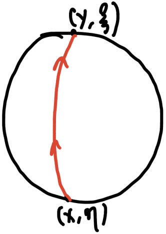
- It is simply connected
- Any geodesic has no conjugate points
- $\partial M$ is strictly convex

Some features that simple manifolds pose are :

- Any two points of a simple manifold can be joined by a unique distance minimizing geodesic depending smoothly on the endpoints
- There are no trapped geodesics.

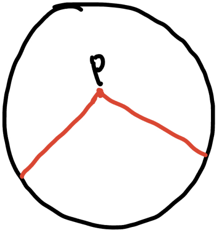
# Measurements on the boundary

Differentiating $d(x, y)$ gives the scattering relation $(x, \eta) \mapsto (y, \xi)$.




However, scattering relation still does not provide any information about the interior of $M$.

We study geodesics that reflect at some interior point $p \in M$.



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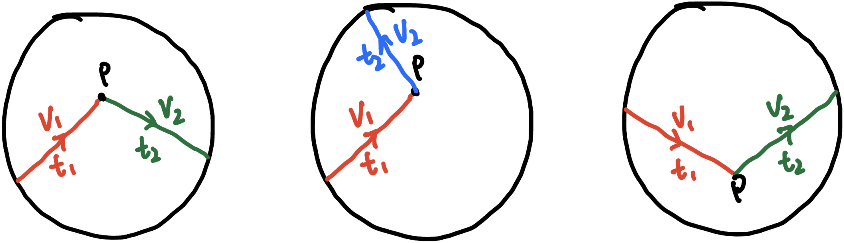
# Broken scattering relations

In a broken scattering relation $v_1 \mathcal{B}_T v_2$, we know the entering direction $v_1$ and exiting direction $v_2$ of a broken geodesic and the total travel time $T = t_1 + t_2$. We do not know the exact locations of $p \in M$.




The family $\{\mathcal{B}_T : \ T > 0\}$ of relations is called the broken scattering relations of Riemannian manifold $(M, g)$.

# Known results

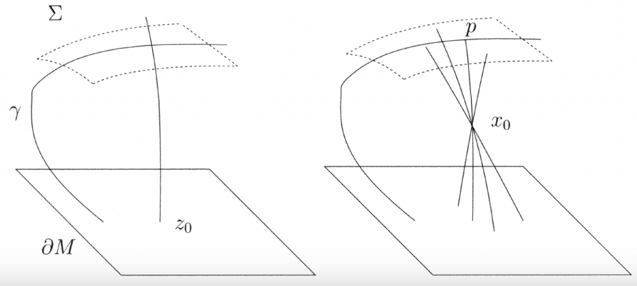
**Theorem (Kurylev-Lassas-Uhlmann, 2010)**

*Let $(M, g)$ be a compact connected Riemannian manifold with a nonempty boundary of dimension $n \geq 3$. Then $\partial M$ and $\mathcal{B}_T$ determine the isometry type of the manifold uniquely.*

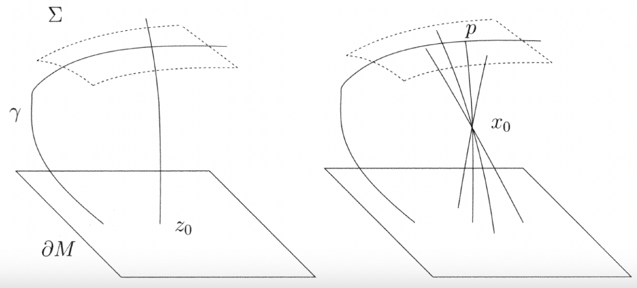
- de Hoop-Ilmavirta-Lassas-Saksala, 2021 : An analogous result on foliated and reversible Finsler manifolds ($n \geq 3$)

# Sketch of proof

The crucial step is to reduce broken scattering relations to travel time via a construction of focusing surface.




Let $z_0 \in \partial M$ and let $U$ be a neighborhood of $z_0$. Define $\Psi : U \times \mathbb{R}_+ \to M$ by $\Psi(z,t) = \exp_z(t\xi(z))$, where $\xi : U \to SU$ is given by $\gamma_{z,\xi(z)}(t(z)) = x_0$. Then $\Psi$ is a local diffeomorphism.




Let $\Sigma$ be an $(n-1)$-dimensional submanifold of $M$ that contains part of the geodesic $\gamma$, and let $\tilde{U} \subset U$ be a neighborhood of $z_0$. Then $\tilde{\Psi} : \tilde{U} \to \tilde{\Psi}(\tilde{U}) \subset \gamma$, where $\tilde{\Psi}(z) = \Psi(z, t(z))$, is a diffeomorphism of $(n-1)$-dimensional manifolds. This is a contradiction if $n \geq 3$, but not when $n = 2$.

# Uniqueness of the broken scattering relations on simple manifolds

Every simple Riemannian manifold is diffeomorphic to the closed unit ball $\mathbb{D}^n$ of $\mathbb{R}^n$. Thus, from here onwards we study simple metrics on $\mathbb{D}^n$.

**Theorem (Ilmavirta-L.-Saksala, 2022)**

*Let $n \geq 2$, and let $g_1$ and $g_2$ be two simple Riemannian metrics. If the broken scattering relations of $g_1$ and $g_2$ coincide, then there exists a smooth Riemannian isometry $\Psi \colon (\mathbb{D}^n, g_1) \to (\mathbb{D}^n, g_2)$ whose boundary restriction $\Psi \colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is the identity map.*

Similar to Kurylev, Lassas, and Uhlmann, the key step of the proof is a reduction to travel time data.

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# Travel time data

For every point $p \in M$ its travel time function $r_p : \partial M \to \mathbb{R}$ is defined by the formula

$$r_p(z) = d(p, z).$$

The location of point sources $p$ are unknown.

The travel time map of the Riemannian manifold $(M, g)$ is then given by the formula

$$\mathcal{R} : (M, g) \to (C(\partial M), \| \cdot \|_\infty)$$

with $\mathcal{R}(p) = r_p$.

The image set $\mathcal{R}(M) \subset C(\partial M)$ of the travel time map is called the travel time data of the Riemannian manifold $(M, g)$.

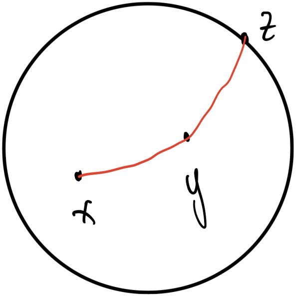
# $\mathcal{R}$ is a metric isometry

- $\mathcal{R}$ is 1-Lipchitz : By triangle inequality,

$$|r_x(z) - r_y(z)| = |d(x,z) - d(y,z)| \leq d(x,y)$$

- For any $x, y \in \mathbb{D}^n$, there exists a unique distance minimzing geodesic connecting $x$ and $y$ to some $z \in \mathbb{S}^{n-1}$. Then

$$|r_x(z) - r_y(z)| = |d(x,z) - d(y,z)| = d(x,y).$$



# Distance of travel time data

$\mathcal{R}(\mathbb{D}^n)$ is a compact subset of the Banach space $C(\mathbb{S}^{n-1}, \|\cdot\|_\infty)$.

We set the distance of travel time data of two simple Riemannian metrics $g_1$ and $g_2$ on $\mathbb{D}^n$ to be

$$d_H^{C(\mathbb{S}^{n-1})}(\mathcal{R}_1(\mathbb{D}^n), \mathcal{R}_2(\mathbb{D}^n)) \geq 0,$$

where $d_H$ is the Hausdorff distance.

Moreover, we say that the travel time data of the simple Riemannian metrics $g_1$ and $g_2$ on $\mathbb{D}^n$ coincide if

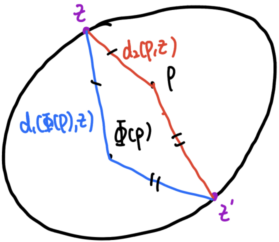
$$d_H^{C(\mathbb{S}^{n-1})}(\mathcal{R}_1(\mathbb{D}^n), \mathcal{R}_2(\mathbb{D}^n)) = 0.$$

# Travel time data is invariant under boundary fixing diffeomorphism

If $\Phi\colon \mathbb{D}^n \to \mathbb{D}^n$ is a diffeomorphism whose restriction on $\mathbb{S}^{n-1}$ is the identity map and $g_1$ is any simple metric of $\mathbb{D}^n$, then the pullback metric $g_2 := \Phi^* g_1$ is a simple metric on $\mathbb{D}^n$ isometric to $g_1$. Thus the equality

$$d_2(p, z) = d_1(\Phi(p), \Phi(z)) = d_1(\Phi(p), z)$$

(valid for all $p \in \mathbb{D}^n$ and $z \in \mathbb{S}^{n-1}$) yields the equations $\mathcal{R}_2(\mathbb{D}^n) = \mathcal{R}_1(\mathbb{D}^n)$ and $d_H^{C(\mathbb{S}^{n-1})}(\mathcal{R}_1(\mathbb{D}^n), \mathcal{R}_2(\mathbb{D}^n)) = 0$.



# Earlier results

Uniqueness :

- Katchalov-Kurylev-Lassas, 2001 : $\mathcal{R}(M)$ determines the isometry class of any compact, connected, oriented, and smooth Riemannian manifold
- de Hoop-Ilmavirta-Lassas-Saksala, 2021 : $\mathcal{R}(M)$ determines a Finsler metric up to a natural obstruction in the direction of the tangent bundle corresponding to distance minimizing geodesics that reach the boundary.
- de Hoop-Ilmavirta-Lassas-Saksala, 2021 : $\mathcal{R}(M)$ from multiple sources determines the isometry of a Riemannian manifold
- Pavlechko-Saksala, 2022 : Partial travel time data determines a compact manifold with strictly convex boundary up to isometry

Stability :

- Katsuda-Kurylev-Lassas, 2007 : Hölder type stability under certain geometric bounds

# Gromov-Hausdorff distance

To measure how close two compact metric spaces $X$ and $Y$ are to each other, we use the Gromov–Hausdorff distance

$$
\begin{aligned}
d_{GH}(X,Y) := \inf\{ & d_H^Z(f(X), g(Y)); \\
& Z \text{ is a metric space}, \\
& f\colon X \to Z \text{ and } g\colon Y \to Z \\
& \text{ are isometric embeddings}\}.
\end{aligned}
$$

$d_{GH}(X,Y) = 0$ if and only if the metric spaces $X$ and $Y$ are isometric.

# Lipschitz stability of the travel time data

**Theorem (Ilmavirta-L.-Saksala, 2022)**

*Let $n \geq 2$, and let $g_1$ and $g_2$ be two simple Riemannian metrics of $\mathbb{D}^n$. Then*

$$d_{GH}((\mathbb{D}^n, g_1), (\mathbb{D}^n, g_2)) \leq d_H^{C(\mathbb{S}^{n-1})}(\mathcal{R}_1(\mathbb{D}^n), \mathcal{R}_2(\mathbb{D}^n)).$$

*In particular, if the travel time data for two metrics coincide, then they agree up to a boundary fixing isometry.*

# Meyers-Steenrod Theorem

A key component of the proof is the following result :

**Theorem (Myers-Steenrod, 1939)**

*Every distance-preserving map between two connected Riemannian manifolds is a smooth isometry of Riemannian manifolds.*

# Sketch of proof of travel time data stability

- $\mathcal{R} : \mathbb{D}^n \to C(\mathbb{S}^{n-1})$ is an isometry.
- If $\mathcal{R}_2(\mathbb{D}^n) = \mathcal{R}_1(\mathbb{D}^n)$, then

$$\Psi := \mathcal{R}_2^{-1} \circ \mathcal{R}_1 : (\mathbb{D}^n, d_1) \to (\mathbb{D}^n, d_2)$$

  is a well-defined bijective metric isometry. By Myers-Steenrod theorem, $\Psi$ is a smooth Riemannian isometry.
- Claim of the theorem folllows by using $f = \mathcal{R}_1$, $g = \mathcal{R}_2$, and $Z = C(\mathbb{S}^{n-1})$ in the definition of Gromov-Hausdorff distance.

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**Theorem (Ilmavirta-L.-Saksala, 2022)**

*Let $n \geq 2$, and let $g_1$ and $g_2$ be two simple Riemannian metrics. If the broken scattering relations of $g_1$ and $g_2$ coincide, then there exists a smooth Riemannian isometry $\Psi\colon (\mathbb{D}^n, g_1) \to (\mathbb{D}^n, g_2)$ whose boundary restriction $\Psi\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is the identity map.*

# Idea of proof : reduction step to travel time data

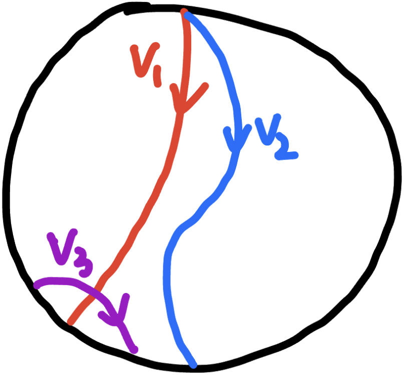
**Proposition (Ilmavirta-L.-Saksala, 2022)**

*Let $g_1$ and $g_2$ be two simple Riemannian metrics on $\mathbb{D}^n$ whose first fundamental forms agree on $\mathbb{S}^{n-1}$. If the broken scattering relations of these metric coincide, then their travel time data also agree.*

From $v_1 \mathcal{B}_T v_2$, we need to recover (1) the scattering relation and (2) the travel times $t_1$ and $t_2$ such that $t_1 + t_2 = T$.

Suppose that $g$ is a simple Riemannian metric of $\mathbb{D}^n$.

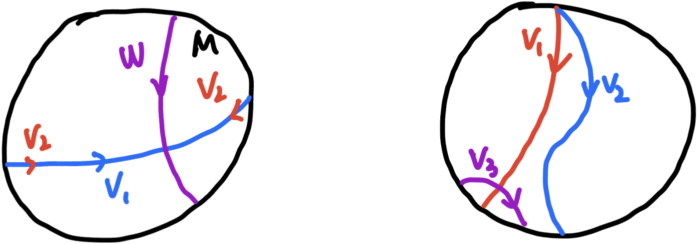
- Suppose $\gamma_{v_1}$ and $\gamma_{v_2}$ are two geodesics that do not have exactly the same endpoints on the boundary. There exists another geodesic $\gamma_{v_3}$ that intersects $\gamma_{v_1}$ but not $\gamma_{v_2}$.



- Let $v_1, v_2 \in \partial_{\text{in}} S\mathbb{D}^n$. The following two statements are equivalent :
  (1) We have $V(v_1) = V(v_2)$, where

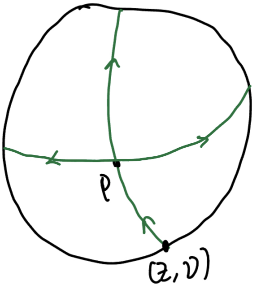
$$V(v_i) := \{w \in B\mathbb{S}^{n-1} : \text{there is } T > 0 \text{ for which } v_i \mathcal{B}_T w\}.$$

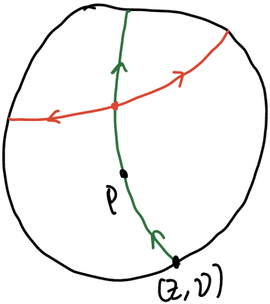
  (2) Either $v_1 = v_2$ or $v_2 = -\phi_{\tau_{\text{exit}}(v_1)}(v_1)$.




- The broken scattering relations determine the scattering relation and exit time function on $\partial_{\text{in}} S\mathbb{D}^n$.

To recover travel times, let $z \in \partial M$ and consider a geodesic normal to the boundary $\gamma_\nu$. Let $p = \gamma_\nu(t), 0 < t < \tau_{\text{exit}}(\gamma_\nu)$.



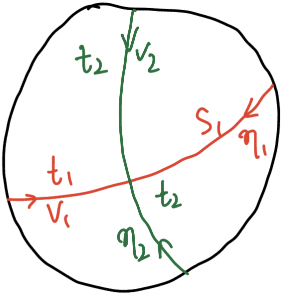



- Suppose that $v_1 \mathcal{B}_T v_2$. Since $g$ is simple, then $\gamma_{v_1}$ and $\gamma_{v_2}$ intersect exactly once. Then there are some numbers $t_1, t_2, s_1, s_2 \geq 0$ that satisfy the equations

$$t_1 + t_2 = T(v_1, v_2), \qquad t_1 + s_1 = T(v_1, \eta_1),$$
$$t_2 + s_2 = T(v_2, \eta_2), \qquad t_1 + s_2 = T(v_1, \eta_2).$$

Then

$$t_1 = \frac{1}{2}\left(T(v_1, v_2) - T(v_2, \eta_2) + T(v_1, \eta_2)\right) \quad \text{and}$$
$$t_2 = T(v_1, v_2) - t_1.$$



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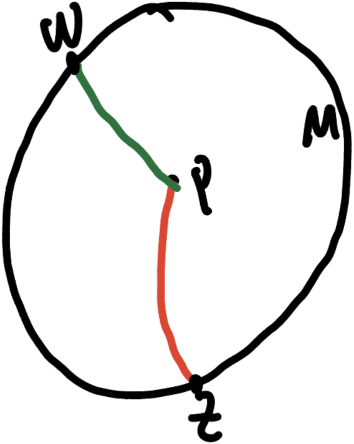
# Travel time is difficult to measure

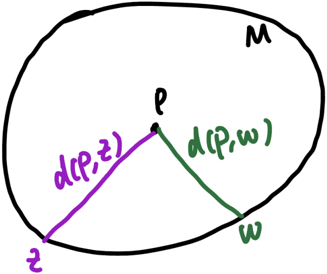
If the origin time of a seismic event is unknown, then the travel time is difficult to measure.




$d(p, z)$ = travel time ($p \to z$) = arrival time ($p \to z$) - origin time

$d(p, z) - d(p, w)$ = arrival time ($p \to z$) - arrival time ($p \to w$)

# Travel time difference data




The travel time difference function of a point $p \in M$ is the function

$$D_p \colon \partial M \times \partial M \to \mathbb{R},$$
$$D_p(z, w) = d(p, z) - d(p, w).$$

Then the travel time difference map and the travel time difference data of the Riemannian manifold $(M, g)$ are

$$\mathcal{D} \colon (M, g) \to (C(\partial M \times \partial M), \| \cdot \|_\infty)$$

with $\mathcal{D}(p) = \frac{1}{2} D_p$, and its image set

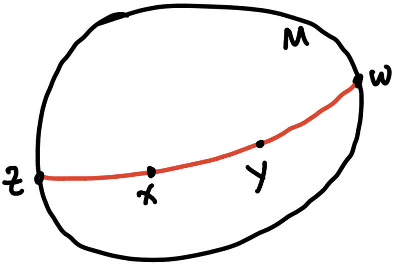
$$\mathcal{D}(M) \subset C(\partial M \times \partial M),$$

respectively.

# $\mathcal{D}$ is a metric isometry

- $\mathcal{D}$ is 1-Lipchitz.
- For any $x, y \in \mathbb{D}^n$, there exists a unique globally distance minimizing geodesic $\gamma$ that goes through $x$ and $y$, having some endpoints $z, w \in \mathbb{S}^{n-1}$. We have

$$|(\mathcal{D}(x)-\mathcal{D}(y))(z,w)| = \frac{1}{2}|d(x,z)-d(y,z)+d(y,w)-d(x,w)| = d(x,y)$$



# Distance of travel time difference data

We set the distance of the travel time difference data of two simple Riemannian metrics $g_1$ and $g_2$ on $\mathbb{D}^n$ to be

$$d_H^{C(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}(\mathcal{D}_1(\mathbb{D}^n), \mathcal{D}_2(\mathbb{D}^n)) \geq 0,$$

where $d_H^{C(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}$ is the Hausdorff distance of the Banach space $(C(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}), \|\cdot\|_\infty)$.

Moreover, we say that the travel time difference data of the simple Riemannian metrics $g_1$ and $g_2$ on $\mathbb{D}^n$ coincide if

$$d_H^{C(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}(\mathcal{D}_1(\mathbb{D}^n), \mathcal{D}_2(\mathbb{D}^n)) = 0.$$

The travel time difference data is invariant under boundary fixing diffeomorphism.

# Earlier results

Uniqueness :

- Lassas-Saksala, 2019 : $D_p(z, w)$ was measured for every $p \in N$ between any $z, w \in F$, where $F \subset N$ contains an open subset of a closed manifold $(N, g)$. Then the metric on $F$, together with the travel time difference data, determine $(N, g)$ up to isometry.
- de Hoop-Saksala, 2019 : Travel time difference data measured on the $\partial M$ determines $(M, g)$ up to isometry.
- Ivanov, 2022 : A complete Riemannian manifold with boundary is uniquely determined, up to an isometry, by its distance difference representation on the boundary.

Stability :

**Theorem (Ivanov, 2020)**

*Suppose $(M_1, g_1)$ and $(M_2, g_2)$ are n-dimensional Riemannian manifolds with $n \geq 2$ and satisfy certain geometric bounds. Assume that $M_1 \cap M_2 = F \neq \emptyset$ is open, they induce the same topology and the same differential structure on $F$, and $g_1|_F = g_2|_F$ . Assume that $F$ contains a geodesic ball of radius $\rho_0$.Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d_H(\mathcal{D}_F^1(M_1), \mathcal{D}_F^2(M_2)) < \delta$ implies $d_{GH}(M_1, M_2) < \varepsilon$.*

However, this stability result does not have a modulo of continuity.

# Lipschitz stability of the travel time difference data

**Theorem (Ilmavirta-L.-Saksala, 2022)**

*Let $n \geq 2$, and let $g_1$ and $g_2$ be two simple Riemannian metrics of $\mathbb{D}^n$. Then*

$$d_{GH}((\mathbb{D}^n, g_1), (\mathbb{D}^n, g_2)) \leq d_H^{C(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}(\mathcal{D}_1(\mathbb{D}^n), \mathcal{D}_2(\mathbb{D}^n)).$$

*In particular, if the travel time difference data for two metrics coincide, then they agree up to a boundary fixing isometry.*

# Sketch of proof

- $\mathcal{D} : \mathbb{D}^n \to C(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ is an isometry.
- If $\mathcal{D}_2(\mathbb{D}^n) = \mathcal{D}_1(\mathbb{D}^n)$, then

$$\Psi := \mathcal{D}_2^{-1} \circ \mathcal{D}_1 : (\mathbb{D}^n, d_1) \to (\mathbb{D}^n, d_2)$$

  is a well-defined bijective metric isometry. By Myers-Steenrod theorem, $\Psi$ is a smooth Riemannian isometry.
- Claim of the theorem follows by using $f = \mathcal{D}_1$, $g = \mathcal{D}_2$, and $Z = C(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ in the definition of Gromov-Hausdorff distance.

Thank you very much for your attention !