The Lorentzian scattering rigidity problem and rigidity of stationary metrics

Plamen Stefanov

Purdue University

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The Riemannian case

The **scattering/lens rigidity** problem:

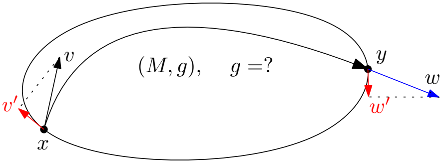


Figure: The scattering relation $\mathcal{S} : (x, v) \mapsto (y, w)$

The lens relation: $(\mathcal{S}, \ell)$, where $\ell(x, v)$ is the travel time.
A better way to do it is to project v, w to $T_x(\partial M)$, $T_y(\partial M)$, and think of $\mathcal{S}$ and ℓ as

$$\mathcal{S} : (x, v') \mapsto (y, w'), \quad \ell : (x, v') \mapsto \mathbb{R}_+ \cup \infty.$$

Definition 1

(M, g) is scattering rigid, if $\mathcal{S}$ determines (M, g) uniquely, up to an isometry ψ fixing ∂M pointwise. In other words, $\mathcal{S} = \hat{\mathcal{S}}$ implies $g = \psi^* \hat{g}$.

Lens rigid: similarly but we use $(\mathcal{S}, \ell)$ as data.

The **boundary rigidity** problem: We use the boundary distance function as data.

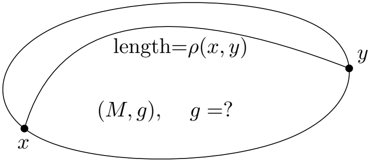


Figure: The boundary distance function $\rho(x,y)$. Same as $\ell(x,v)$ but parameterized differently.

It maps, locally, $\rho:\mathbb{R}^{2n-2}\mapsto\mathbb{R}$, while $\mathcal{S}:\mathbb{R}^{2n-2}\mapsto\mathbb{R}^{2n-2}$. In fact, $\mathcal{S}$ is a symplectic map and it is the canonical map of the hyperbolic DN map associated with g (which is an FIO). When x, y are not conjugate along a geodesic connecting them, we can define ρ locally, and then ρ is the generating function of $\mathcal{S}$.

$$\mathcal{S}:(x,-\operatorname{grad}'_x\rho)\mapsto(y,\operatorname{grad}'_y\rho).$$

Here and below, primes denote tangential projections. Then ρ determines $(\mathcal{S},\ell)$!

The boundary distance function ρ has another advantage. If we linearize $g \mapsto \rho_g$ near a background g_0, we get the nice looking X-ray transform of two-tensor fields

$$Xf(\gamma) = \int f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt$$

over maximal geodesics γ, where $f = \delta g$.

If you linearize $g \mapsto \mathcal{S}_g$ instead, we get an uglier X-ray transform involving ∇f, f and weights. It can be analyzed but it is not fun.

But. . . we had to assume non-conjugacy. . . Not a big deal. Knowing the scattering relation, we can determine the jet of g at ∂M, then extend g smoothly in a known way, and push y a bit along that geodesic. Then x and y would not be conjugate along it. The "distance" $\rho(x, y)$ may maximize, not minimize length but this is not a problem. More about it — later.

There are a lot of works on Riemannian lens/scattering/boundary rigidity. Herglotz and Wiechert and Zoeppritz, Mukhometov, Romanov, Michel, Croke, Sharafutdinov, Pestov-Uhlmann, S-Uhlmann, S-Uhlmann-Vasy, etc.

It is known that in a neighborhood of an open dense set of simple metrics, including simple real analytic ones, we have local uniqueness (up to the gauge) and stability.

It is known that under a foliation condition, we have boundary rigidity. In particular, non-positive curvature (and simple connectedness) or non-negative curvature is a sufficient condition, plus a strictly convex boundary. We even have local uniqueness near a strictly convex boundary point.

The linear problem has been studied a lot as well, by the same authors and others, including Monard. Under the conditions above, we have injectivity and stability.

Partial data problems have been studied and understood. Also, the contribution of conjugate points on the lost of stability has been studied.

(M, g) Lorentzian, $x \in U \subset \partial M$, $v \in T_x U$ lightlike, future pointing. Shoot and measure when $\gamma_{x,v}$ hits $V \subset \partial M$ again.

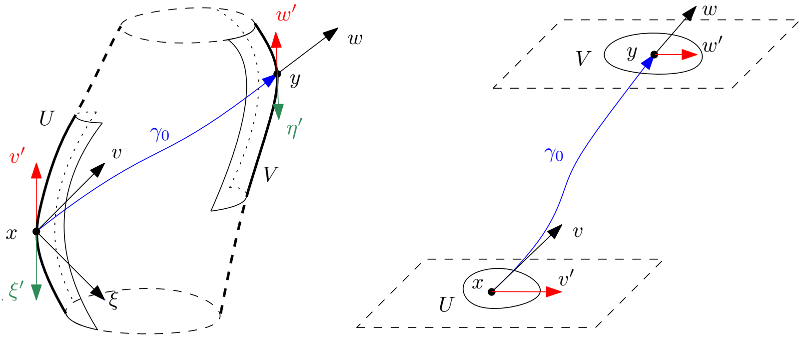


Figure: The scattering relation $\mathcal{S} : (x, v') \to (y, w')$. Left: timelike to timelike; Right: spacelike to spacelike.

As before, it is better to identify v with its projection v', same for w.

Small problem: there is no "natural" parametrization of lightlike geodesics $\gamma_{x,v}(s)$. We can rescale s to ks, $k > 0$ and it is not worse or better than before.

No big deal, we just think of $\mathcal{S}(x, v')$ as homogeneous of order 1 in its second argument.

It is useful to lift $\mathcal{S}$ to the cotangent bundle, call that map $\mathcal{S}^\sharp$. We proved with YANG YANG that $\mathcal{S}^\sharp$ is the canonical relation of the DN map related to $\Box_g u = 0$, so it is symplectic.

Question

Is there an analog $r(x, y)$ of the boundary distance function $\rho(x, y)$ in the Lorentzian case?

We want

- r and $\mathcal{S}$ to determine each other.
- r to be the generating function of $\mathcal{S}$.
- The linearization of $g \mapsto r_g$ near a background g_0 to give us the light ray X-ray transform

$$Lf(\gamma) = \int f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\,dt$$

  over lightlike geodesics with $f = \delta g$.

Why? This gives some hope to solve the inverse problem by linearization. Problem: L misses timelike singularities (moving faster than light).

Defining function

It is problematic to use Lorentzian distance: it is actually zero when x and y can be reached by (locally unique) lighlike geodesic. There is a separation function which is singular, and does not seem to help either.

With the setup as in the picture above, we assume

Non-conjugacy assumption (NC)

x_0 and y_0 are not conjugate along γ_0.

We start with

Definition 2

(a) The set $\Sigma \subset U \times V$ consists of pairs (x, y) so that x and y are connected by a unique lightlike geodesic (locally).

(b) The smooth function $r : U \times V \to \mathbb{R}$ is called a defining function of Σ, if (i) $r = 0$, $dr \neq 0$ on Σ, and (ii) $r(x, y) < 0$ if and only if the locally unique geodesic connecting x and y is timelike.

The latter is just a sign convention. Of course, r is not unique.

Theorem 3

With the assumptions above, with the non-conjugate assumption (NC),
(a) Σ is a codimension one submanifold of $U \times V$ when U and V are small enough.

Let $r : U \times V \to \mathbb{R}$ be any defining function of Σ. Then

(b) $\{(x, -d'_x r, y, d'_y r);\ r(x,y) = 0\}$ coincides locally with the graph of some reduced representation of $\mathcal{S}^\sharp$.

(c) If g_τ is an one-parameter family of Lorentzian metrics smoothly depending on τ near $\tau = 0$, so that $g_0 = g$, and r_τ are associated defining functions smoothly depending on τ, we have

$$\frac{d}{d\tau}\Big|_{\tau=0} r_\tau(x,y) = \kappa(x,y) \int_0^1 \big\langle f, \dot{\gamma}_{[x,y]}(t) \otimes \dot{\gamma}_{[x,y]}(t) \big\rangle \, dt \quad \text{on } \Sigma = \{r(x,y) = 0\},$$

where $f = dg_\tau / d\tau|_{\tau=0}$, $[0,1] \ni t \to \gamma_{[x,y]}(t)$ is the locally unique lightlike geodesic in the metric g connecting x and y, and κ is a smooth non-vanishing function.

Remarks

The non-uniqueness of the defining function r due to the freedom to multiply by any elliptic factor $\kappa(x,y)$ implies the following. If $r_\kappa = \kappa r$ is another defining function, then in (a), $\mathcal{S}^\sharp(x,v') = (y,w')$ with $v' = -d'_x r$, $w' = d'_y r$ is replaced by $\mathcal{S}^\sharp(x,\kappa v') = (y,\kappa w')$, which is just another reduced representation of $\mathcal{S}^\sharp$. Each one of them determines $\mathcal{S}^\sharp$ by homogeneity. Replacing r_τ by $\kappa_\tau r_\tau$ multiplies the right-hand side by κ_0.

The non-trivial elliptic factor κ is inevitable since the light ray transform on the right is determined up to rescaling of the parameter t, and the defining function is determined by such a factor as well.

The graph of $\mathcal{S}^\sharp$ (locally) coincides with the twisted conormal bundle $N'\Sigma \setminus 0$. This already follows from S-Yang since under (NC), the Schwartz kernel of the DN Map is conormal to Σ. We get this here directly.

Corollary 4

The scattering relation $\mathcal{S}$ determines r uniquely up to an elliptic factor. On the other hand, each defining function r determines $\mathcal{S}$ uniquely.

The introduction of $r(x,y)$ legitimizes the interest in L.

The proof is constructive. We construct one such $r(x, y)$ (every other one is obtained by multiplication by $\kappa \neq 0$) using the concept of energy. For a smooth curve $[a, b] \ni t \to c(t)$, one defines the (non positive or negative definite) energy functional

$$E(c) = \frac{1}{2} \int_a^b (\dot{c}(t), \dot{c}(t))_g \, dt.$$

Each geodesic (lightlike or not) connecting the endpoints is a critical point of $E(c)$ without being a minimizer or a maximizer.

Take g_τ; and $t \in [0, 1] \mapsto \gamma_\tau(t)$ connecting x and y. Then

$$\frac{d}{d\tau}\Big|_{\tau=0} E_{g_\tau}(\gamma_\tau) = \frac{1}{2} \int_0^1 \left\langle f, \dot{\gamma}_0 \otimes \dot{\gamma}_0 \right\rangle dt, \quad f := \frac{d}{d\tau}\Big|_{\tau=0} g_\tau.$$

Set

$$r(x, y) = E(\gamma_{[x,y]}), \quad (x, y) \in U \times V,$$

where $[0, 1] \ni t \to \gamma_{[x,y]}(t)$ is the unique (locally) geodesic connecting x and y. It is one choice of a defining function.

Example

Take a Riemannian metric $h(x)$ on M, with a product with $-dt^2$:

$$g = -dt^2 + h(x).$$

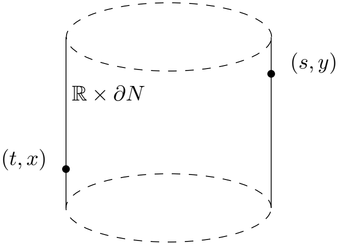
We have a full separation of variables, no need of time-space formulation. Let $\rho(x, y)$ be the boundary distance function associated with h. Then we can take

$$r((t, x), (s, y)) = t - s + \rho(x, y).$$

We have $r = 0$ iff (t, x), (s, y) are connected by a lightlike geodesic. We can also take

$$r_1((t, x), (s, y)) = -(t - s)^2 + \rho^2(x, y).$$

The defining function given by the energy is $r_2 = r_1/(s - t)$.



Gauge invariance (known)

Invariance under diffeomorphisms ψ a local diffeomorphism near γ_0, fixing U and V pointwise. Then $\mathcal{S}_g = \mathcal{S}_{\psi_* g}$.

Linearizing this, we get $L(d^s v) = 0$ for every covector field v vanishing on U and V, where d^s is the symmetrized differential.

Invariance under conformal changes Let $\tilde{g} = c(x)g$, where $c(x) > 0$ is a smooth function. The factor c just changes the parameterization of lighlike geodesics (so that they do not satisfy the geodesic equation anymore) but $\mathcal{S}$ is preserved: $\mathcal{S}_g = \mathcal{S}_{cg}$.

Linearizing this, we get $L(cg) = 0$ for every c which is obvious by itself since $cg(\dot\gamma, \dot\gamma) = 0$ since $g(\dot\gamma, \dot\gamma) = 0$.

Scattering rigidity and light ray transform injectivity formulations

Scattering Rigidity

$$\mathcal{S}_{g_1} = \mathcal{S}_{g_2} \Longleftrightarrow g_2 = c\psi^* g_1 \text{ with } c > 0$$

S-injectivity

$$Lf = 0 \Longleftrightarrow f = d^s v + \lambda g \text{ with } v = 0 \text{ on } U \cup V$$

Now when we know that L is an actual linearization of $\mathcal{L}$ (of r, actually), what do we know about its s-injectivity?

Well, it has a microlocal "blind spot" — cannot see timelike singularities. Stable inversion is not possible unless we assume additional structure, like in Vasy and Wang.

- Injectivity on functions $f(t,x)$ with $f=0$ for $|x|>R$, with g Minkowski, was proved by S in 1988 using the partial analyticity of $\hat{f}(\tau,\xi)$ in ξ.
- Extended to g analytic under a convex foliation condition by S; local results; using analytic microlocal analysis.
- Extended by RabieniaHaratbar to one-forms, g Minkowski.
- Lassas, Oksanen, Uhlmann and S.: microlocal analysis for Lorentzian metrics on functions; also on 2-tensors for expanding metrics.
- Y. Wang and Vasy and Wang.
- Feizmohammadi, Ilmavirta, Oksanen: results for stationary metrics. In particular, injectivity up to the gauge for 2-tensors of compact support in some (including the Minkowski) stationary cases.

Stationary metrics

Stationary metrics:

$$g = -\lambda(x)dt^2 + 2\tilde{\omega}_j(x)dt\,dx^j + \tilde{h}_{ij}(x)dx^i dx^j.$$

with $\lambda > 0$. The coefficients are t-independent. In matrix form,

$$g = \begin{pmatrix} -\lambda & \tilde{\omega}_1 & \dots & \tilde{\omega}_n \\ \tilde{\omega}_1 & \tilde{h}_{11} & \dots & \tilde{h}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\omega}_n & \tilde{h}_{n1} & \dots & \tilde{h}_{nn} \end{pmatrix}.$$

It is convenient to write (complete the square):

$$g = \lambda(x)\Big(- \Big(dt + \omega_j(x)dx^j\Big)^2 + h_{ij}(x)dx^i dx^j\Big).$$

There are three objects here: λ, the 1-form ω and the $n \times n$ metric h ($\dim M = 1 + n$). We write $g = g_{\lambda,\omega,h}$.

This form can be derived from global assumptions of global hyperbolicity and existence of a complete Killing field $\partial/\partial t$.

A natural manfiold M would be a cylinder $M = \mathbb{R}_t \times N_x$.
We want to identify all "natural" transformations preserving $\mathcal{S}$, and the stationary structure.

- Diffeomorphisms in the x variable (only) fixing the boundary $\psi : N \to N$. Set $\Psi = Id \times \psi$; then $\mathcal{S}_{\Psi^* g} = \mathcal{S}_g$.
- Let $\phi(x)$ vanish on ∂N. Set $\Phi(t, x) = (t + \phi(x), x)$. Those are x-dependent time shifts. They fix ∂M pointwise as well. Then $\Phi_* g$ has the same form but with ω replaced by $\omega + d\phi$.

The latter says something interesting. One may think that there is a "natural" time variable t, say up to a constant shift. In fact, that shift can be x-dependent.

Definition 5

The metrics $g_{\lambda,\omega,h}$ and $g_{\hat{\lambda},\hat{\omega},\hat{h}}$ are called gauge equivalent, if there exists a diffeomorphism $\psi : N \to N$ fixing ∂N pointwise, and a function ϕ vanishing on ∂N, so that

$$\hat{\omega} = \psi^*(\omega + d\phi), \quad \hat{h} = \psi^* h.$$

We want to show that $\mathcal{S}_g = \mathcal{S}_{\hat{g}}$ implies that g and $\hat{g}$ are gauge equivalent.

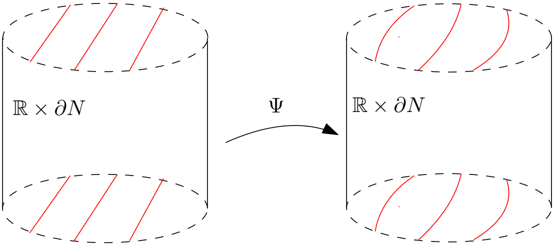


Figure: A space deformation.

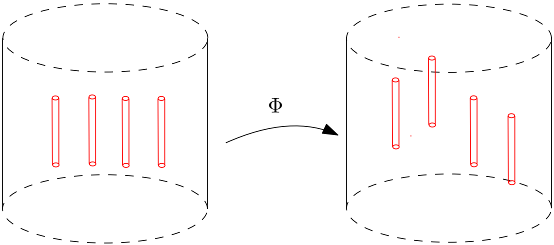


Figure: An x-dependent time shift.

Prior results:

- Feizmohammadi, Ilmavirta, L. Oksanen: they project the dynamical system onto the base N, and get a certain dynamical system there. Then they derive results about the linear problem (about L) depending on the behavior of that system.
- Uhlmann, Yang, Zhou: they use the time separation function (taking into account timelike geodesics as well), under some more restrictive technical assumptions: the nonlinear problem.

Recall

$$g = \lambda(x)\Big(- \Big(dt + \omega_j(x)dx^j\Big)^2 + h_{ij}(x)dx^i dx^j\Big).$$

We can take $\lambda = 1$ (by gauge invariance).

We project $M = \mathbb{R} \times N$ on the "base" N as well. This is not an orthogonal projection! It is identifying points lying on the same time-line.

$$g = -\left(dt + \omega_j(x)dx^j\right)^2 + h_{ij}(x)dx^i dx^j, \quad (\lambda = 1).$$

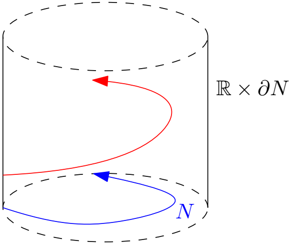


Figure: A projection on the base.

It turns out that the projection onto the base is a magnetic system! If you think about it, what else can it be[1]: we have a Riemannian metric and an one-form... The actual proof was provided by A. GERMINARIO, 2007.

Rigidity for magnetic systems was studied by DAIRBEKOV, PATERNAIN, UHLMANN AND S. Every results there translates into a result for stationary metrics, as we show.

[1] © M. Zworski

What is a magnetic system?

On (N, h), we are given a closed two-form Ω. We can assume topology allowing us to claim $\Omega = d\omega$, and that ω would be the form in the stationary metric. We can associate $Y : TN \to TN$, which basically $d\omega$ with one index raised, by $\langle \Omega, u \otimes v \rangle = (Yu, v)_h$. The dynamics is

$$D_s \dot{\gamma} = Y \dot{\gamma}.$$

One can think of the r.h.s. as a Lorentz force orthogonal to $\dot{\gamma}$ (since Ω is anti-symmetric) representing the action of a magnetic field. As an example, if h is Euclidean and Y is constant, one gets space spirals.

The speed $|\dot{\gamma}|$ is preserved along the flow. We fix it to be one. If we change the initial speed, we get different curves in general.

Let $\ell_{x,y}$ be the travel time. The natural boundary data is not the travel time, it is the action

$$\mathbb{A}(x, y) = \ell_{x,y} - \int_{\gamma_{[x,y]}} \omega.$$

In particular, it is invariant under adding $d\phi$ to ω.

Gauge equivalent magnetic systems

$$\hat{h} = \psi^* h, \quad \hat{\omega} = \psi^* \omega + d\phi.$$

[Dairbekov, Paternain, Uhlmann and S], 2007.

- A linearization of $\mathbb{A}(x, y)$ is the following X-ray transform

$$I[f,\beta](\gamma) = \int \langle f, \dot{\gamma} \otimes \dot{\gamma} \rangle + \int \langle \beta, \dot{\gamma} \rangle,$$

  where $f = \frac{1}{2}\delta g$, $\beta = -\delta\omega$.

- $\mathbb{A}$ determines $\mathcal{S}_{\mathrm{mag}}$ and vice versa.

Results about linearized magnetic systems

S-injectivity means $f = d^s v$, $\beta = d\phi - Yv$. True in those cases:

(i) with an explicit bound of the curvature, following the energy method going back to Mukhometov, Romanov, Pestov and Sharafutdinov;

(ii) in a given conformal class,

(iii) for analytic ones using analytic microlocal analysis,

(iv) locally, near generic ones using the analytic result.

Gauge equivalence holds in those cases:

(i) Two-dimensional (simple) magnetic systems are boundary rigid. This was derived generalizing the Riemannian result by Pestov and Uhlmann, without a linearization.

(ii) If $\hat{g} = \mu g$ with $\mu > 0$ a function, then equality of the lens data implies $\mu = 1$ and $\hat{\omega}$ is gauge equivalent to ω.

(iii) Real analytic simple magnetic systems with the same lens data are gauge equivalent. This follows from a boundary determination of the jets of h and ω, and then by analytic continuation.

(iv) Generic local rigidity near simple magnetic systems with s-injective linearizations.

Back to stationary metrics

Write $\gamma(s) = (t(s), x(s))$. Then the geodesic system becomes triangular:

$$\dot{t} + \langle \omega, \dot{x} \rangle = k,$$
$$D_s \dot{x} = kY\dot{x},$$

with k constant along every null geodesic. The second one is a magnetic geodesic equation, while the first one, integrated, tells us that t is an action variable, not a time one!

one of the main theorems, not formulated formally here says that

- Each of the quantities: $\mathcal{S}$ (on $M = \mathbb{R} \times N$), $\mathcal{S}_{\mathrm{mag}}$ and $\mathbb{A}$ (the latter two on N) determine each other in an explicit way.
- In particular, knowing $\mathcal{S}$, we know $\mathbb{A}$, and all magnetic results hold.

We write $g = g_{\lambda,\omega,h}$.

Theorem 6

Let (M, g) and $(M, \hat{g})$ be simple and stationary. Then $\mathcal{S} = \hat{\mathcal{S}}$ implies that g and $\hat{g}$ are gauge equivalent if and only if $\mathbb{A} = \hat{\mathbb{A}}$ implies that (h, ω) and $(\hat{h}, \hat{\omega})$ are magnetically gauge equivalent. In particular, the simple magnetic system (N, h, ω) is boundary rigid if and only if the stationary Lorentzian manifold $(M, g_{\lambda,\omega,h})$ is boundary rigid.

Corollary 7 (Rigidity in $1 + 2$ dimensions)

Simple stationary manifolds (M, g) of dimension $\dim M = 1 + 2$ *are lens rigid.*

Corollary 8 (Rigidity in a given conformal class)

Let (M, g) and $(M, \hat{g})$ be two simple stationary Lorentzian systems so that $\hat{h} = \mu(x)h$ with some $\mu > 0$. If $\mathcal{S} = \hat{\mathcal{S}}$, then $\mu = 1$ and $\hat{\omega} = \omega + d\phi(x)$ with some ϕ vanishing on ∂N.

Corollary 9 (Generic local rigidity)

There exists an open dense set $\mathcal{G}^k$ so that for every $(h_0, \omega_0) \in \mathcal{G}^k$, there exists $\varepsilon > 0$ such that for every two simple stationary metrics $g = g_{\lambda,\omega,h}$ and $\hat{g} = g_{\hat{\lambda},\hat{\omega},\hat{h}}$ for each of which (h, α), $(\hat{h}, \hat{\alpha})$ is an ε close to (h_0, α_0) in $C^k(N)$, we have the following:

$$\mathcal{S} = \hat{\mathcal{S}}$$

implies that $\hat{g}$ and g are gauge equivalent.