On the determination of polyhedral interfaces from boundary measurements

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Long term project

Dream

Find one or more polyhedral inclusions in an elastic medium by taking a finite number of boundary measurements

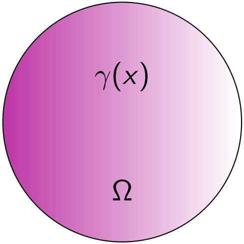
Reality

Lipschitz stable determination of a polyhedral inclusion in a conducting body from the Dirichlet-to-Neumann map

Inverse conductivity problem

Recover $\gamma > 0$ defined in Ω from boundary values of solutions of the equation

$$\operatorname{div}(\gamma \nabla u) = 0 \quad \text{in } \Omega.$$



Boundary measurements are encoded in the **Dirichlet to Neumann map**:

$$\begin{aligned} \Lambda_\gamma : H^{1/2}(\partial\Omega) &\rightarrow H^{-1/2}(\partial\Omega) \\ f &\rightarrow \gamma \frac{\partial u}{\partial \nu} \end{aligned}$$

where u solves

$$\begin{cases} \operatorname{div}(\gamma \nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

The forward map is the nonlinear map

$$F:\gamma\in L^{\infty}(\Omega)\to\Lambda_{\gamma}\in\mathcal{L}\left(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega)\right)$$

1. **Uniqueness** of the solution of the inverse problem: injectivity of F.

$$F(\gamma_0)=F(\gamma_1)\overset{?}{\Rightarrow}\gamma_0=\gamma_1$$

   Are the measurements enough to distinguish between two different coefficients?

2. **Stability** for the inverse problem: continuity of F^{-1}.

$$\|\gamma_0-\gamma_1\|\overset{?}{\leq}\omega\left(\|F(\gamma_0)-F(\gamma_1)\|\right)$$

   for $\omega(t)\to 0$ as $t\to 0$.

$$F(\gamma_0) = F(\gamma_1) \overset{?}{\Rightarrow} \gamma_0 = \gamma_1$$

Isotropic conductivities

- $n \geq 2$, γ_0, γ_1 piecewise analytic Kohn Vogelius (1984, 1985)
- $n \geq 3$ and $\gamma_0, \gamma_1 \in C^2(\overline{\Omega})$ Sylvester Uhlmann (1987), $W^{1,\infty}(\Omega)$ Caro Rogers (2016), $W^{1,3}(\Omega)$ Haberman (2015)
- $n = 2$ and $\gamma_0, \gamma_1 \in W^{2,p}(\Omega)$ Nachman (1995), Brown Uhlmann (1997) , $\gamma_0, \gamma_1 \in L^\infty(\Omega)$ Astala Paivairinta (2006)

The inverse conductivity problem is unstable, but stability can be restored by restricting the space of unknowns adding a priori information.

- Alessandrini (1988) $n \geq 3$, $\|\gamma\|_{W^{2,\infty}(\Omega)} \leq E$
- Barcelo, Faraco, Ruiz (2007) $n = 2$, $\|\gamma\|_{C^\alpha(\overline{\Omega})} \leq E$.

$$\|\gamma_0 - \gamma_1\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star).$$

- Clop, Faraco, Ruiz (2010) $n = 2$, $\|\gamma\|_{W^{\alpha,p}(\Omega)} \leq E$, $\alpha > 0$

$$\|\gamma_0 - \gamma_1\|_{L^2(\Omega)} \leq \omega(\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star)$$

- Caro, García, Reyes (2013) $n \geq 3$, $\gamma \in C^{1,\epsilon}(\overline{\Omega}) \leq E$

$$\|\gamma_0 - \gamma_1\|_{C^{0,\delta}(\Omega)} \leq \omega(\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star)$$

In all these results $\omega(t) = C|\log t|^{-\eta}$.

A limit to stability with information on regularity

Mandache (2001) has proved that logarithmic stability is the best possible stability using as a priori assumption of the form

$$\|\gamma\|_{C^k(\overline{\Omega})} \leq E, \quad \forall k = 0, 1, 2....$$

Strategy:

Look for **a priori assumptions** on conductivity

- physically relevant
- give rise to better stability (**Lipschitz stability**)

Assume $F : K \subset L^\infty \to \mathcal{L}$, K subset of a finite dimensional space
unknown conductivity depends on finitely many parameters
For example

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j \chi_{D_j}(x).$$

- Reasonable for most applications, e.g. medical imaging (different tissues), geophysical prospection (different rocks, layers of the earth), nondestructive testing of materials (composite materials)

$F : \gamma \in K \subset L^\infty \to \Lambda_\gamma \in \mathcal{L}$, where K denotes compact subset of a finite dimensional space.

Steps to prove Lipschitz stability estimate:

1. prove that F is injective (uniqueness);
2. prove that F is differentiable and evaluate the Frechét derivative DF;
3. prove that DF is continuous and bounded from below.

The bound from below of the derivative DF gives a bound from above of the constant C in the stability estimate

$$\|\gamma_0 - \gamma_1\| \le C\|F(\gamma_0) - F(\gamma_1)\| = C\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star$$

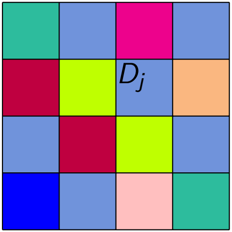
Bacchelli, Vessella (2006)
Bourgeois (2013)

Parameter identification

$$\gamma = \sum_{j=1}^{N} \gamma^j \chi_{D_j}, \quad \bigcup_{j=1}^{N} \overline{D}_j = \Omega \subset \mathbb{R}^n$$

known domains D_j unknown parameters γ^j

$$K = \left\{ \sum_{j=1}^{N} \gamma^j \chi_{D_j} : \alpha_0^{-1} \leq \gamma^j \leq \alpha_0 \right\}$$



Theorem (Alessandrini, Vessella (2005))

$$\sum_{j=1}^{N} |\gamma_0^j - \gamma_1^j| \leq C \|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_\star$$

where C depends only on α_0 and on the known domains $D_1, \ldots, D_N$.

Remark: a local Dirichlet to Neumann map is enough for this result.

Other PDEs and systems

- Complex valued coefficients γ

  Beretta, F. (2011)

- Elasticity

  Beretta, F.,Vessella (2014)

  Beretta, F., Morassi, Rosset, Vessella (2014)

  Beretta, de Hoop, F., Vessella, Zhai (2017)

- Anisotropic equation

  Gaburro, Sincich (2015), Foschiatti, Gaburro Sincich (2021), Foschiatti (2024)

- Helmholtz equation $\Delta u + qu = 0$

  Beretta, de Hoop, Qiu (2013)

  Beretta, de Hoop, Faucher, Scherzer (2016)

- Piecewise linear coefficients

  Alessandrini, de Hoop, Gaburro, Sincich (2016 and 2017)

Finite number of measurements

Theorem

Let X and Y be Banach spaces. Let $A \subset X$ be an open subset, $W \subset X$ be a finite dimensional subspace and $K \subset W \cap A$ be a compact and convex subset. Let $F \in C^1(A, Y)$ be a Lipschitz map satisfying the Lipschitz stability estimate

$$\|x_1 - x_2\|_X \leq C\|F(x_1) - F(x_2)\|_Y \quad \forall x_1, x_2 \in K,$$

for some $C > 0$.
Let $Q_N : Y \to Y$ be bounded linear maps for $N \in \mathbb{N}$ that approximates the identity, then

$$\|x_1 - x_2\|_X \leq 2C\|Q_N F(x_1) - Q_N F(x_2)\|_Y \quad \forall x_1, x_2 \in K.$$

Alberti, Santacesaria (2022)

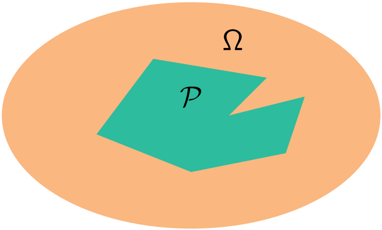
For finitely many electrodes — Harrach (2017)

Interface or inclusion identification

A polygonal inclusion in the conductivity problem

$$\begin{cases} \operatorname{div}(\gamma \nabla u) &= 0 \text{ in } \Omega \subset \mathbb{R}^2, \\ u &= f \text{ on } \partial\Omega. \end{cases}$$

$$\gamma = 1 + (k-1)\chi_P$$



Inverse problem:

is it possible to stably to recover P from the DN-map Λ_γ?

If instead of a polygon we have a smooth inclusion, **logarithmic stability** is optimal and has been proved by Alessandrini-Di Cristo (2005)
If $\mathcal{P}$ is a polygon then we show that **Lipschitz stability** holds

Finite measurements

- Barcelo, Fabes, Seo (1994) uniqueness with one special measure for convex polyhedra
- Seo (1996) uniqueness with two measurements for polygons
- Kim, Yamamoto (2004) uniqueness for convex polygons
- Bellout, Friedman, Isakov (1992) local stability for convex polygons
- Hanke **(just released on arxiv)** Lipschitz stability of a nonlinear inverse conductivity problem with two Cauchy data pairs (polygons)

The class of non degenerate polygons

Let $\mathfrak{D}$ be the set of closed, simply connected, simple polygons $\mathcal{P} \subset \Omega$ such that:

$$\mathcal{P} \text{ has at most } N_0 \text{ sides each one with length greater than } d_0, \qquad (1)$$

$$\partial\mathcal{P} \text{ is of Lipschitz class with constants } r_0 \text{ and } K_0, \qquad (2)$$

there exists a constant $\beta_0 \in (0, \pi/2]$ such that the angle β in each vertex of $\mathcal{P}$ satisfies the conditions

$$\beta_0 \leq \beta \leq 2\pi - \beta_0 \text{ and } |\beta - \pi| \geq \beta_0, \qquad (3)$$

and

$$dist(\mathcal{P}, \partial\Omega) \geq d_0. \qquad (4)$$

N_0, d_0, r_0, K_0, and β_0 are our **a priori data**

Notice that we do not assume convexity of the polygon.

Theorem

Let $\mathcal{P}^0, \mathcal{P}^1 \in \mathfrak{D}$ and let

$$\gamma_0 = 1 + (k-1)\chi_{\mathcal{P}^0} \text{ and } \gamma_1 = 1 + (k-1)\chi_{\mathcal{P}^1}.$$

There exists a positive constant C depending only on the a priori data such that,

$$d_H\left(\partial\mathcal{P}^0, \partial\mathcal{P}^1\right) \leq C\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_*$$

and

$$\|\gamma_0 - \gamma_1\|_{L^1(\Omega)} \leq C\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_*$$

Beretta, F. (2022)

Remark: a local Dirichlet to Neumann map is enough for this result.

1. Take advantage of a logarithmic stability estimate

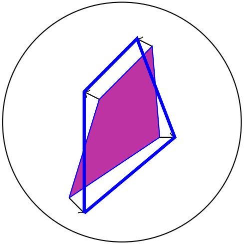
$$\|\gamma_1 - \gamma_0\|_{L^2(\Omega)} \leq C \left|\log \|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_*\right|^{-\alpha},$$

Clop, Faraco, Ruiz, (2010)

2. Use a priori assumptions to show that if $\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_* \leq \epsilon_0$, then $\mathcal{P}^0$ and $\mathcal{P}^1$ have the same number N of vertices P_j^0 and P_j^1 ($j = 1, \dots, N$) that can be ordered so that

$$dist\left(P_j^0, P_j^1\right) \leq C \left|\log \|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_*\right|^{-\alpha}$$

3. Set $P_j^t = P_j^0 + t(P_j^1 - P_j^0)$.
Define $\mathcal{P}^t$ the polygon of vertices P_j^t for $j = 1, \dots, N$
Set $\gamma_t = 1 + (k-1)\chi_{\mathcal{P}^t}$



4. The map $F(t, f, g) =< \Lambda_{\gamma_t}(f), g >$ is differentiable and, at $t = 0$,

$$\frac{d}{dt} < F(\gamma_t)f, g >_{|_{t=0}} = (k-1)\int_{\partial\mathcal{P}^0} (M_0 \nabla u_0^e \cdot \nabla v_0^e)(\Phi_0 \cdot n_0)$$

where $u_0^e = u_0|_{\Omega\setminus\mathcal{P}^0}$ and $v_0^e = v_0|_{\Omega\setminus\mathcal{P}^0}$

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla u_0) = 0 \text{ in } \Omega, \\ u_0 = f \text{ on } \partial\Omega. \end{cases} \qquad \begin{cases} \operatorname{div}(\gamma_0 \nabla v_0) = 0 \text{ in } \Omega, \\ v_0 = g \text{ on } \partial\Omega. \end{cases}$$

$M_0 = \tau_0 \otimes \tau_0 + \frac{1}{k} n_0 \otimes n_0$ where τ_0 and n_0 are the tangent and outer normal directions on $\partial\mathcal{P}^0$ and Φ_0 is a piecewise affine map such that

$$\Phi_0(P_j^0) = P_j^1 - P_j^0, \text{ for } j = 1, 2, \ldots, N$$

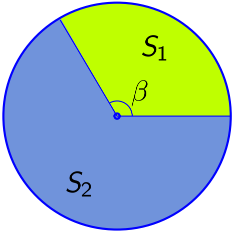
BERETTA-F.-VESSELLA (2017)
HANKE ARXIV (2024)

Regularity estimates for solutions of equations and systems with discontinuous coefficients

$$\operatorname{div}\left(\left(1+(k-1)\chi_{S_1}\right)\nabla u\right)=0 \text{ in } B_R(0)$$

Set

$$u_j = u_{|_{S_j}}.$$

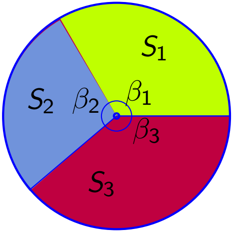


There exist $\omega > 1/2$ and C depending only on k, R and β such that

$$|\nabla u_j(x,y)| \leq C\|u\|_{L^2(B_R(0))}(x^2+y^2)^{\frac{\omega-1}{2}}$$

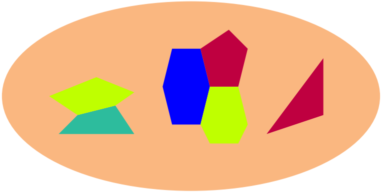
Bellout, Friedman, Isakov (1992)

Singularity at vertices with 3 regular sectors



The singularity of the gradient is integrable on sides of the partition if $\overline{\beta} \leq \beta_j \leq \pi - \overline{\beta}$ and $\overline{\gamma} \leq \gamma_j \leq \overline{\gamma}^{-1}$

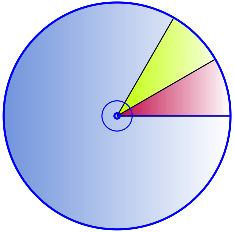
$$|\nabla u_j(x,y)| \leq C\|u\|_{L^2(B_R(0))}(x^2+y^2)^{\frac{\omega-1}{2}}$$

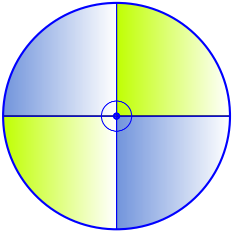


Beretta-F.-Vessella (2018)

Non integrable singularities

3 sectors with $\beta_2 \geq \pi$





4 regular sectors

Strategy of the proof: bound from below the derivative

Tools

(I) **Quantitative estimates of unique continuation**

(II) Construction of **singular solutions** for equations with discontinuous coefficients and study of their asymptotic behaviour near discontinuity interfaces.

(Druskin (1982))

Strategy of the proof: bound from below the derivative

Bound from below is the quantitative counterpart of **injectivity** of the derivative.

$$\frac{d}{dt}F(t,f,g)_{|_{t=0}} = 0 \text{ for every } f,g \quad \Rightarrow \quad \partial\mathcal{P}^0 = \partial\mathcal{P}^1$$

That is:

$$\int_{\partial\mathcal{P}^0} (M_0 \nabla u_0^e \cdot \nabla v_0^e)(\Phi_0 \cdot n_0) = 0$$

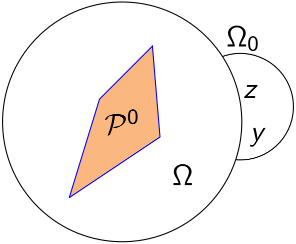
for every u_0, v_0 such that $\operatorname{div}(\gamma_0 \nabla u_0) = \operatorname{div}(\gamma_0 \nabla v_0) = 0$ in Ω implies

$$\partial\mathcal{P}^0 = \partial\mathcal{P}^1.$$

Assume

$$\int_{\partial\mathcal{P}^0} (M_0 \nabla u_0^e \cdot \nabla v_0^e)(\Phi_0 \cdot n_0) = 0$$

for every u_0, v_0 such that $\mathrm{div}\,(\gamma_0 \nabla u_0) = \mathrm{div}\,(\gamma_0 \nabla v_0) = 0$ in Ω.

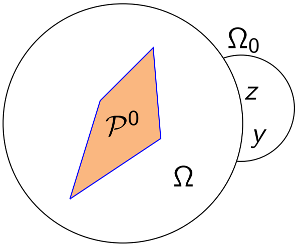


- Extend Ω to a larger domain Ω_0.
- Choose $u_0(x) = G_0(x, y)$ and $v_0(x) = G_0(x, z)$, where G_0 is the Green's function in Ω_0
- Define the function

$$S(y, z) = \int_{\partial\mathcal{P}^0} (M_0 \nabla G_0^e(\cdot, y) \cdot \nabla G_0^e(\cdot, z))(\Phi_0 \cdot n_0)$$

for $y, z \in \Omega \setminus \partial\mathcal{P}^0$

$$S(y,z) = \int_{\partial\mathcal{P}^0} (M_0 \nabla G_0^e(\cdot,y) \cdot \nabla G_0^e(\cdot,z))(\Phi_0 \cdot n_0)$$



- For $y,z \in \Omega_0 \setminus \Omega$, $G_0(\cdot,y)$ and $G_0(\cdot,z)$, solve $\operatorname{div}(\gamma_0 \nabla G_0) = 0$ in Ω, hence
  $$\mathbf{S} \equiv \mathbf{0} \text{ in } \Omega_0 \setminus \Omega$$
- S is **harmonic** with respect to both y and z in $\Omega_0 \setminus \mathcal{P}^0$
- UCP $\Rightarrow$ S is **zero** for $y,z \in \Omega_0 \setminus \mathcal{P}^0$,
- Asympotics of Green function $\Rightarrow$ $M_0 \nabla G_0^e(\cdot,y) \cdot \nabla G_0^e(\cdot,z)$ **diverges** as y,z tends to $\partial\mathcal{P}^0$
- $\Rightarrow$ $\Phi_0 \cdot n_0 = 0 \Rightarrow \partial\mathcal{P}^0 = \partial\mathcal{P}^1$.

Finite measurements

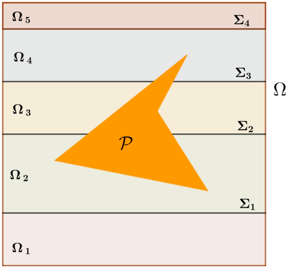
Theorem

Let Δ^2 be the family of non degenerate triangles contained on Ω with positive distance from the boundary and let $\gamma_T = 1 + (k-1)\chi_T$ for $T \in \Delta^2$.
Let $Q_N : \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)) \to \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$ be bounded linear maps for $N \in \mathbb{N}$ that approximates the identity, then

$$\|\gamma_{T_1} - \gamma_{T_2}\|_{L^1(\Omega)} \leq C\|Q_N(\Lambda_{\gamma_{T_1}}) - Q_N(\Lambda_{\gamma_{T_2}})\|_*$$

Alberti, Arroyo, Santacesaria (2023)

An inclusion in a stratified background



Assume that $\mathcal{P}$ is non degenerate and for every vertex P_j of $\mathcal{P}$ we have

$$dist(P_j, \Sigma_i) \geq d_0,$$

The stratification of the background is known

The main issue in this case is the lack of integrability of the gradient where the sides meet the interface.

$$\gamma_0 = 1 + \sum_{j=1}^{M}(\gamma^j - 1)\chi_{D^j}, \qquad \Phi_t(x) = x + tV(x),$$

$$D_t^j = \Phi_t(D^j), \qquad \gamma_t = 1 + \sum_{j=1}^{M}(\gamma^j - 1)\chi_{D_t^j}$$

$$\frac{d}{dt} < \Lambda_{\gamma_t} f, g >_{|t=0} = -\int_\Omega \gamma_0 \mathcal{A} \nabla u_0 \nabla v_0 dx$$

where

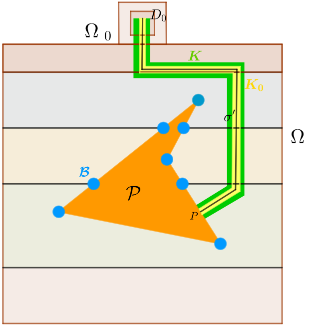
$$\mathcal{A} = \mathrm{div}(V)I - (\nabla V + \nabla^T V)$$

and $\mathrm{div}(\gamma_0 \nabla u_0) = \mathrm{div}(\gamma_0 \nabla v_0) = 0$ in Ω, $u_0 = f$ and $v_0 = g$ on $\partial\Omega$.

Beretta, Micheletti, Perotto, Santacesaria (2018)

In the case of a single polygon, integration by parts give

$$-\int_{\Omega}\gamma_0\mathcal{A}\nabla u_0\nabla v_0 dx=(k-1)\int_{\partial\mathcal{P}^0}(M_0\nabla u_0^e\cdot\nabla v_0^e)(\Phi_0\cdot n_0)$$

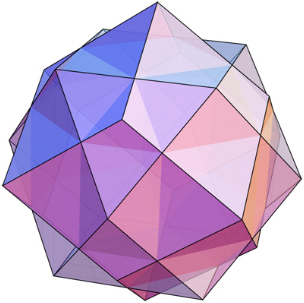


Hence we can go from one form to the other at least on subsets far from "forbidden points"

There exists C depending only on the a priori data such that

$$d_H\left(\partial\mathcal{P}^0, \partial\mathcal{P}^1\right) \leq C\|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_*.$$

Beretta, F. , Vessella (2021)



- Geometric description is more involved
- No rough stability estimate available
- Behaviour of ∇u close to vertices not given

Class of regular polyhedra

A polyhedron $D \subset \Omega$ is in $\mathfrak{D} = \mathfrak{D}(r_0, R_0, \theta_0, M_0)$ if

Strict Inclusion:

$$dist(D, \partial\Omega) \geq r_0.$$

Dihedral non-degeneracy: the angle α at each edge of D satisfies

$$\alpha \in (\theta_0, \pi - \theta_0) \cup (\pi + \theta_0, 2\pi - \theta_0).$$

Face non-degeneracy: for any face F^D there exists $x_0 \in F^D$ such that

$$B'_{r_0}(x_0) \subset F^D,$$

Edge non-degeneracy: for each edge σ_{ij}^D of D

$$length\left(\sigma_{ij}^D\right) \geq r_0.$$

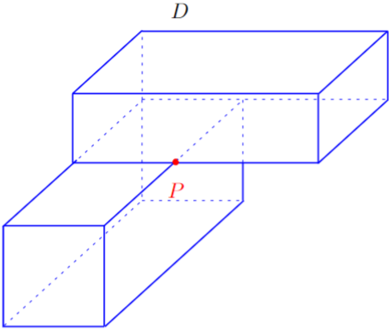
Face angle non-degeneracy: for each internal angle β of each face

$$\beta \in (\theta_0, \pi - \theta_0) \cup (\pi + \theta_0, 2\pi - \theta_0).$$

Lipschitz regularity $\Omega \setminus D$ is connected and has Lipschitz boundary with constants r_0 and M_0.

Remarks

- The number of vertices , edges and faces of a polyhedron in $\mathfrak{D}$ is bounded from above by a constant N_0 depending only on r_0, R_0, and M_0.
- Recall that Lipschitz regularity of $\Omega \setminus D$ is not implied by the previous assumptions.



Theorem

Let D_0, D_1 be two polyhedral inclusions in $\mathfrak{D}$. Let 1 and k be the conductivity coefficients of $\Omega \setminus D_i$ and D_i, for $i = 0, 1$, respectively. If, for some ε with $0 < \varepsilon < 1$,

$$\left\| \Lambda_{D_0}^{\Sigma} - \Lambda_{D_1}^{\Sigma} \right\|_{\star} \leq \varepsilon,$$

then

$$d_H(\partial D_0, \partial D_1) \leq \widetilde{\omega}(\varepsilon),$$

where $\widetilde{\omega}(\varepsilon)$ is an increasing function in $[0, +\infty)$ such that

$$\widetilde{\omega}(t) \leq C |\log t|^{-\zeta}, \qquad \text{for all } 0 < t < 1,$$

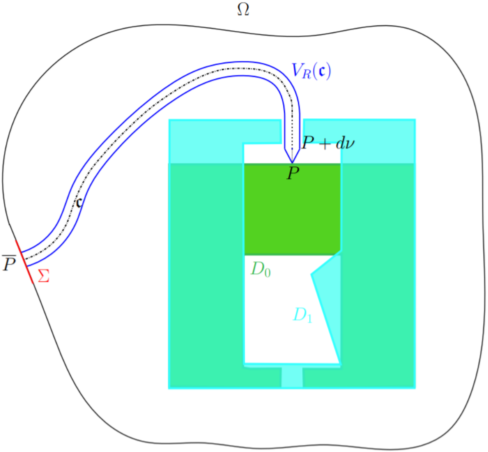
where $C > 0$ and ζ, $0 < \zeta \leq 1$ are constants depending only on the a priori data.

Aspri, Beretta, F., Vessella (2022)

Alessandrini Identity

$$\left\langle (\Lambda^{\Sigma}_{\gamma_{D_0}} - \Lambda^{\Sigma}_{\gamma_{D_1}}) u_0\big\lfloor_{\Sigma}, u_1\big\lfloor_{\Sigma} \right\rangle = \int_{\Omega} (k-1)(\chi_{D_0} - \chi_{D_1}) \nabla u_0 \cdot \nabla u_1 \, dx,$$

where χ_{D_i}, for $i = 0, 1$, is the characteristic function of D_i.



For smooth inclusions

Alessandrini, Di Cristo, Morassi, Rosset (2014)

For the description of close polyhedra

L. Rondi (2008)

Distances of polyhedra

Proposition

There exist two positive constants δ_0 and C depending only on the a priori data, such that, if for some D_0 and D_1 in $\mathfrak{D}$,

$$d_H\left(\partial D_0, \partial D_1\right) \leq \delta_0,$$

then D_0 and D_1 have the same number N of vertices $\left\{V_i^{D_0}\right\}_{i=1}^N$ and $\left\{V_i^{D_1}\right\}_{i=1}^N$, respectively, which can be ordered in such a way that

$$dist\left(V_i^{D_0}, V_i^{D_1}\right) \leq C d_H\left(\partial D_0, \partial D_1\right).$$

Moreover, for each edge or face in D_0 there is an edge or a face in D_1 with corresponding vertices.

Let $D_t = \Phi_t(D_0)$ Given $f, g \in H^{\frac{1}{2}}_{co}(\Sigma)$, let u_t and v_t be solution corresponding to the inclusion D_t with Dirichlet boundary data f and g respectively. Define

$$F(t, f, g) = \left\langle \Lambda^{\Sigma}_{\gamma_{D_t}} f, g \right\rangle = \int_\Omega \gamma_{D_t} \nabla u_t \cdot \nabla v_t \, dx.$$

Distributed derivative

$F(t, f, g)$ is differentiable and

$$F'(0, f, g) = -\int_\Omega \gamma_{D_0} \mathcal{A}_0 \nabla u_0 \cdot \nabla v_0 \, dx$$

where

$$\mathcal{A}_0 = \frac{d}{dt} \left(\left(D\Phi_t^{-1} \right) \left(D\Phi_t^{-1} \right)^T \det \left(D\Phi_t \right) \right) \Big\lfloor_{t=0}.$$

Integration by parts

Boundary representation of the derivative

$$F'(0,f,g) = -\int_{\mathcal{B}} \gamma_{D_0} \mathcal{A}\nabla u_0 \cdot \nabla v_0 \, dx - \int_{\partial\mathcal{B}} \gamma_{D_0} b \cdot \nu \, dx + \int_{\partial D_0 \setminus \mathcal{B}} \mathcal{U} \cdot \nu (k-1) \mathcal{M} \nabla u_0^i \cdot \nabla v_0^i \, dx.$$

$\mathcal{B}$ is a tubular neighborhood of the edge of D_0, b is bounded, $\mathcal{M}$ is the so-called polarization tensor, i.e., a 3×3 matrix with eigenvectors ν and $\nu^\perp$ and with eigenvalues k and 1 and $\mathcal{U}$ is an affine function defined on ∂D_0 such that

$$\mathcal{U}(V_i^{D_0}) = V_i^{D_1} - V_i^{D_0}, \qquad \forall\; i = 1, \dots, N$$

Theorem

Let D_0 and $D_1 \in \mathfrak{D}(r_0, R_0, \theta_0, M_0)$ let $k > 0$, $k \neq 0$ and let Σ be an open portion of $\partial\Omega$ Then, there exists C depending only on the a priori data such that

$$d_H\left(\partial D_0, \partial D_1\right) \leq C \left\| \Lambda^{\Sigma}_{\gamma_{D_0}} - \Lambda^{\Sigma}_{\gamma_{D_1}} \right\|_*$$

Aspri, Beretta, F., Vessella (2022)