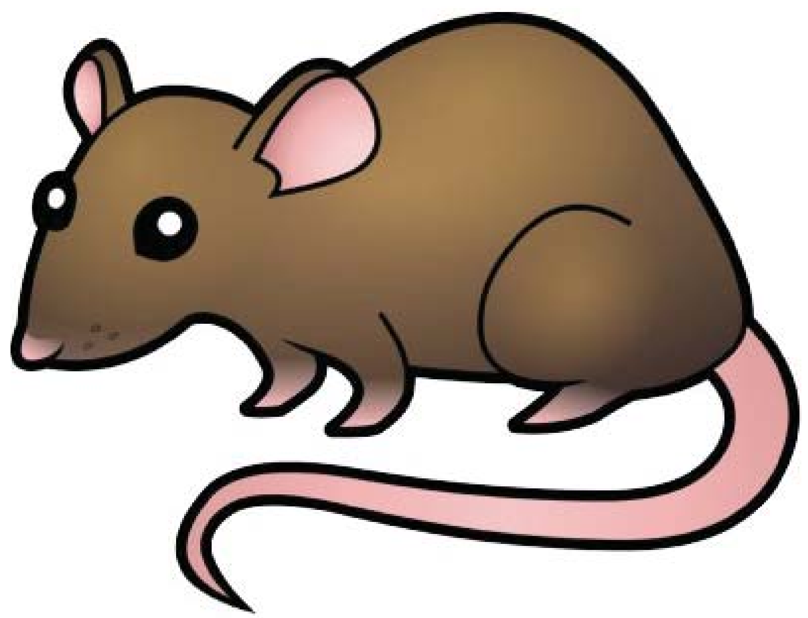
Essential Lessons

- Geometric Sequences
- Geometric Series
- Exponential and Logarithmic Graphs
- Comparing Exponential and Logarithmic Rules
- Solving Exponential Equations Using Logs

References:
1) "Progressions Documents for the Common Core Math Standards." The University of Arizona. Brookhill Foundation, 2007.

2) "Mathematics Framework Chapters." Curriculum Frameworks (CA Dept of Education). 2013.

Math III



Exponential & Logarithmic Functions

Storybook

Standards Addressed

Seeing Structure in Expressions A-SSE
Interpret the structure of expressions. [Polynomial and rational]

1. Interpret expressions that represent a quantity in terms of its context.
b. Interpret complicated expressions by viewing one or more of their parts as a single entity.
2. Use the structure of an expression to identify ways to rewrite it.
Write expressions in equivalent forms to solve problems.
4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

Functions- Interpreting Functions (F-IF)
Interpret functions that arise in applications in terms of the context. [Include rational, square root and cube root; emphasize selection of appropriate models.]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.
7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.
e. Graph exponential and logarithmic functions, showing intercepts and end behavior.
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Unit Story

THE REASONING BEHIND THE FLOW

The unit begins with a scenario about Bacteria that reviews the basics of exponential functions which were first introduced in students' first high school math course (either Algebra 1 or Math 1). Several variations are investigated, varying the starting amount and varying the factor by which the population grows. Students are asked to find when the population will reach a certain number, which will raise awareness in the students that they do not yet have the algebraic tools necessary to solve an exponential function and must resort to numerical or graphical methods to find the solution.
Exponential growth can also be referred to as geometric growth, which explains the placement of the next two lessons, Geometric Sequences and Geometric Series, in this unit. Geometric Sequences focuses on helping students connect the various representations of sequences, including :

- A list of terms
- A verbal description
- A recursive formula
- An explicit formula

Connections are made to the idea of an exponential function, while noting the differences in notation used when describing a sequence rather than regular function notation. Sequence notation includes the use of subscripts to identify the term number, which operates similarly to the input of a function. In fact, if we consider the term numbers as inputs then a sequence is a function, usually with a domain restricted to the whole numbers. Students derive and learn to use the formula to find the sum of a finite geometric series.
In Inverse Functions, students identify the need for a new type of function to "undo" an exponential function. This discovery is set in the context of other pairs of functions including linear functions with linear inverses and a quadratic function with a square root inverse. After defining logarithms as the inverses of exponential functions, the lesson Exponential and Logarithmic Graphs looks closely

Formative Assessment

Big Idea #1: What is a geometric sequence? Geometric series?

- There are several good points to assess student understanding of geometric sequences in the lesson Geometric Sequences. At the end of the first page, use an "Ink, Pair, Share" activity. Have students individually write in their own words what defines a geometric sequence. Next have them pair with another student to read aloud what they have written. They can make edits on their own or suggest edits to their partner. Finally, the whole class can work together to create a precise definition using academic vocabulary. Later in the class, students can copy one geometric sequence that they understand well onto a small white board. This could be a sequence from the early part of the lesson or one that they create toward the end. Have students form an inside/outside circle. Ask one question for each pair to discuss then rotate. Questions could be: What is the recursive formula? What is the common ratio? What is a_1? What is the explicit formula?

Big Idea #2: What is an exponential function? What are its characteristics compared to other functions?

- Both of these big ideas can be assessed during the lesson Identifying Key Features of Graphs. Either before or after students have completed the key features pages, give each student a small whiteboard and marker and have them form teams of 3-4. When you call out a key feature, they must each write down a function that has that characteristic. The students then show each other what they have written and discuss which answers are correct. When you call out a function, they must each write down a key feature. Again, they check with each other. Whole class discussions can be held when useful or if teams have disagreements about possible answers.

Big Idea #3:What is a logarithmic fucntion? What are its characteristics compared to other functions?

- Both of these big ideas can be assessed during the lesson Identifying Key Features of Graphs. Either before or after students have completed the key features pages, give each student a small whiteboard and marker and have them form teams of 3-4. When you call out a key feature, they must each write down a function that has that characteristic. The students then show each other what they have written and discuss which answers are correct. When you call out a function, they must each write down a key feature. Again, they check with each other. Whole class discussions can be held when useful or if teams have disagreements about possible answers.

General Strategies For Formative Assessments

Use the following strategies throughout the unit

Thumbs up/down/ sideways

Ask students to rate their understanding. A thumbs up means they understand the topic, thumbs down means they don't and in the middle means they get part of it but still need additional support.

Mini White Boards

Can be used in a variety of ways. Suggestion: Give students problems to solve on white boards and have them raise their boards to show you their answer.

Writing a Summary Statement

Ask students to write a summary of the day's/week's lesson.

Fist to Five

Similar to thumbs up/down, students rate their understanding on a scale of zero (fist) to 5, with 5 being fully understand.

Ticket Out The Door (Exit Ticket)

In the last couple minutes of class, give students a problem or two to complete on a slip of paper. Collect the paper as students are leaving the class.

Sentence Frames & Starters

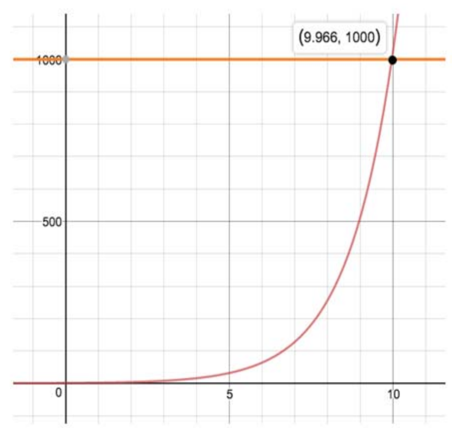
Here are some options to provide to students throughout the activities.

- I agree with _______ because _______.
- I disagree with _______ because _______.
- I did not understand _______________.
- I prefer _______method/ strategy because _______.
- I think that _______.
- What do you mean by _______?
- I think _______ means _____________.

Math Behind the Unit Cont'd & Flow

AND INTENTION OF EACH LESSON

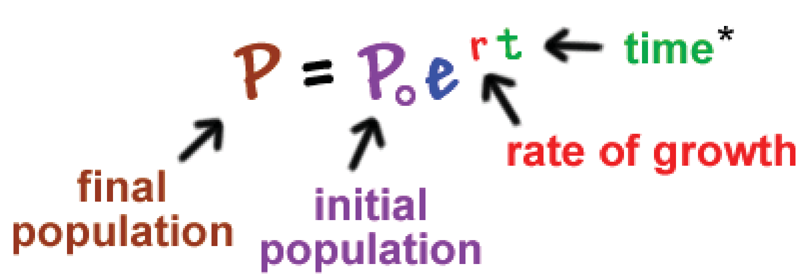
Alternatively, students might use a graphing utility, such as Desmos, to graph both the function, f(x)=2^x, and the horizontal line, y=1000, and find the intersection



These methods can still be used in this course. With the introduction of logarithms, however, students now have access to solving exponential equations using inverse operations, as they typically do with linear, quadratic, rational and radical equations.

Solving multi-step exponential equations using inverse operations still requires selecting the correct order of operations to do in the solving process.

Equation	Operation
$5 \cdot 3^x + 200 = 1000$	Subtract 200 from both sides.
$5 \cdot 3^x = 800$	Divide both sides by 5.
$3^x = 160$	Take the log_3 of both sides.
$x = \log_3 160$	Use a calculator to approximate.
$x \approx 4.620$	



at several pairs of graphs, comparing and contrasting 2^x with $\log_2 x$, 10^x with $\log_{10} x$, and $5 \cdot 3^x$ with $\log_3 {}^{x}/_{5}$. Key features, including domain, range, intercepts and end behaviors are examined. Identifying Key Features of Graphs reprises a lesson structure used in previous courses with representative graphs from all of the key function families studied and described with attention paid to the above key features as well as intervals of increasing or decreasing behavior and average rates of change on intervals.

Each of the standard rules for simplifying expressions with exponents has a matching counterpart in the rules for simplifying expressions with logarithms. These rules are first developed inductively in the lesson Comparing Exponential and Logarithmic Rules. After using numeric and simple variable examples to identify patterns, the rules for simplifying or rewriting logarithmic expressions are identified and justified algebraically.

At various places in the curriculum new types of functions give rise to new types of single variable equations which require new strategies to solve. For example, quadratic functions generate quadratic equations requiring square roots, factoring or the quadratic formula. In this unit, exponential functions generate exponential equations. Solving Exponential Equations is done through the use of logarithms. This lesson allows teachers to work with students to identify which logarithm keys are available on various calculators to find approximate solutions to exponential equations.

The unit ends with a population modeling problems, Rats, that involves both exponential growth and downward pressures on a population. Students initially use dice to simulate the rat population, accounting for both the birth and death rates in the population. The simulation creates a data set that can be modeled using an exponential function. Afterward, analyzing the situation produces an expected growth rate that can be compared to the simulation data and used to predict values of the population at future times and when the population will reach identified values.

Standards Addressed

Building Functions F-BF
Build a function that models a relationship between two quantities. [Include all types of functions studied.]

1. Write a function that describes a relationship between two quantities.
b. Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.

Build new functions from existing functions. [Include simple, radical, rational, and exponential functions; emphasize common effect of each transformation across function types.]

3. Identify the effect on the graph of replacing f(x) by f(x) + k, kf(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.
4. Find inverse functions.
a. Solve an equation of the form f(x)=c for a simple function f that has an inverse and write an expression for the inverse.

Linear, Quadratic, and Exponential Models (F-LE)

4. For exponential models, express as a logarithm the solution to abct = d where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. [Logarithms as solutions for exponentials]
4.1. Prove simple laws of logarithms. CA
4.2 Use the definition of logarithms to translate between logarithms in any base. CA
4.3 Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA

SBAC Claims

- CLAIM #1: CONCEPTS AND PROCEDURES (40% STUDENT SCORE)
- CLAIM #2: PROBLEM SOLVING (20% STUDENT SCORE)
- CLAIM #3: COMMUNICATING AND REASONING (20% STUDENT SCORE)
- CLAIM #4: MODELING AND DATA ANALYSIS (20% STUDENT SCORE)

Types Of Knowledge

- **MEMORIZATION (QUICK RECALL)**
- **PROCEDURAL (FOLLOW STEPS/ DO SOMETHING)**
- **CONCEPTUAL (UNDERSTND BIG IDEA/ EXPLAIN/ DERIVE)**
- **RELATIONAL (APPLY/ ANALYZE/ EVALUATE)**

Unit

WHY RATIONAL FUNCTIONS ARE IMPORTANT

This unit serves the purpose of extending student understanding of exponential functions while introducing logarithmic functions as their inverse. In addition, exponential and logarithmic expressions that are used to define these functions become exponential and logarithmic equations when there is a need to solve for inputs that will generate specific, desired outputs. Solving these equations poses new challenges and requires new techniques.

Where Does

In grade 3, students begin to understand multiplication as repeated addition. This foundation allows upper grade students to see the constant additive growth in a linear function as the multiplicative factor or coefficient of x (slope) in the equation.

In grade 4, students do work looking for and studying patterns with addition and multiplication.

In grade 5, students learn to plot points in the first quadrant on the coordinate grid. They also continue using tables to convert units of measure. The use of an exponent is first introduced in grade 5, but only with a base of 10.

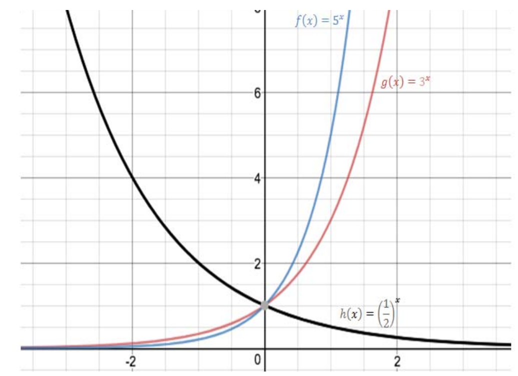
In grade 6, students graph points in all four quadrants in the coordinate plane. Students also collect and graph data, deciding which is the dependent variable and which is the independent variable. This foundation continues to develop into the skills of labeling and scaling a graph as well as interpreting data from a graph. Exponents are used more generally with various bases in grade 6.

Grade 7 has a major focus on proportional relationships, where students collect, graph and analyze data. Students write the equation for a line passing through (0,0) (a proportional relationship), with a focus on understanding the rate (which grade 8 students will call slope or rate of change). The ability to write an equation in the form y=kx is crucial, as students will soon formalize rate of change, study negative rates of change and add initial values to equations.

The Unit!

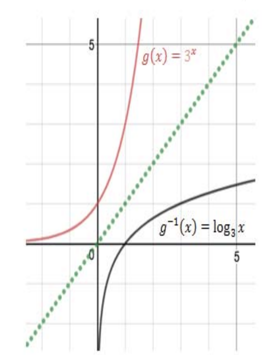
Exponential and Logarithmic Functions

Exponential functions have been studied and used to model situations during the previous two courses in high school. The defining characteristic of the data from an exponential function is that if f is an exponential function, then for any constant c, f(x+c)/f(x) = f(y+c)/f(y) for any values of x and y. This is commonly described as a "common multiplier" or "common ratio" for fixed intervals of change in the input. The general form of an exponential equation, unshifted vertically or horizontally, is $f(x)=a\cdot b^x$. Below are some representative graphs.



An unshifted and unreflected exponential function will approach infinity for one end behavior and will approach y=0 as an asymptote for its other end behavior. Its domain is all real numbers, while its range includes only the positive real numbers.

Logarithmic functions are the inverses of exponential functions. In the graph see here, the logarithmic function is seen as the reflection of the exponential function across the line y=x.



This inverse relationship determines the key features of the unshifted logarithmic function. It has an x-intercept, but no y-intercept. Its domain is only the positive real numbers while its range is all real numbers. It has a vertical asymptote at x=0. And its positive end behavior is to tend to positive infinity, although at smaller and smaller, but still positive, rates of change.

Solving Exponential Equations using Logarithms

Solving a one-variable equation is the process of finding the value or values of that variable that satisfy the equation. The first time students experience exponential functions and the exponential equations that arise from them, they have limited tools to use in solving. For example, if they are working with the function f(x)=2^x, and would like to know when the function reaches the value 1000, they might write the equation, 1000=2^x. In order to solve this equation, students might use a guess and check process to find that 2^{10}=1024 and a calculator to refine their answer to approximately 9.97.

Student Talk Strategies

- Report to a partner
  Each student reports his/ her own answer to a peer. The students listen to their partner's response. ("Turn to a partner on your left." "Now turn to a partner on your right" etc.)
- Give one get one
  After brainstorming ideas, students circulate among other students sharing one idea and getting one. Students fold paper lengthwise they label the left side "give one" and the right side "get one".
- Think, Pair, Share
  Students think about a topic suggested by the teacher. Pairs discuss the topic. Students individually share information from their discussion with the class.
- Inside-outside circle
  Two concentric circles of students stand or sit, facing one another. The teacher poses a question to the class, and the partner responds. At a signal, the outer of inner circle or outer circle rotates and the conversation continues.
- Appointment clock
  Partnering to make future discussion/work appointments.
- Jigsaw
  Group of students assigned a portion of a text; teach that portion to the remainder of the class.

Math Practice Standards

1) Make sense of problems and persevere in solving them.

2) Reason abstractly and quantitatively.

3) Construct viable arguments and critique the reasoning of others.

4) Model with mathematics.

5) Use appropriate tools strategically.

6) Attend to precision.

7) Look for and make use of structure.

8) Look for and express regularity in repeated reasoning.

The Math Behind

Geometric Sequences

Geometric sequences are lists of numbers where the ratio between any two consecutive terms is constant. Consider the following example: 1, 2, 4, 8, 16, ... Looking at the ratio of the consecutive terms, we can see that the ratio remains constant as shown below.

2 ÷ 1 = 2   4 ÷ 2 = 2   8 ÷ 4 = 2   16 ÷ 8 = 2

This constant ratio between the terms is called the common ratio.

Traditionally, sequences are often named using letters from the start of the alphabet and subscript notation is used to identify the term numbers. For example, a_5 would indicate the fifth term of the sequence named "a". Function notation can also be used to identify the terms of a sequence. The table below shows commonly used notation.

Notation	Meaning
n	Term number
a_1 or $a(1)$	First term in a sequence
a_n or $a(n)$	The n-th term in a sequence
a_{n-1} or $a(n-1)$	The term prior to a_n or $a(n)$
d	Common difference
r	Common ratio

Sequences can be described with a formula in two ways, either recursively or explicitly. A recursive formula defines how to get from one term in a sequence to the next. A complete definition for a recursive formula will also need to define the value of one term. The sequence above is defined recursively as follows:

$$a_1=1$$
$$a_{(n+1)}=2\cdot a_n$$

In words, this would mean that the first term is 1 and that the next term of the sequence can be found by multiplying the current term by 2.

This same sequence can be defined explicitly. An explicit formula allows you to find any term number directly rather than needing to know the previous term. The explicit formula for this sequence is: $a_n=2^{(n-1)}$

Geometric Series

Geometric Series are the sums of terms from geometric sequences. Here are some examples:

Finite geometric series: 1+2+4+8+16
Infinite geometric series: 1+2+4+8+16 + . . .

If a geometric series is increasing and infinite, then the value of the series is also infinite. There is however, a formula for the sum of a finite geometric series.

Sum of a Finite Geometric Series: for a series based on a sequence defined by $a_n = a_1 \cdot r^{n-1}$ we have:

$$(a_1 + a_2 + a_3 + a_4 + \cdots + a_n) = \frac{a_{n+1} - a_1}{r - 1}$$

In words, this means that in order to find the sum you must first find the multiplier (common ratio) of the geometric sequence that makes the series. You must also find the next term in the series beyond the one you are adding. Start with this next term and subtract the first term. Next, divide the result by one less than the multiplier

For example, 1+2+4+8+16=(32-1)/(2-1)=31/1=31

Overview

IN INTEGRATED MATH III/ALGEBRA II

Through the course of this unit, students will deepen their understanding of what defines an exponential function and how the graph of an exponential function behaves. In addition, students will study the effects of transformations on the graphs of exponential functions. Finally, students will apply their understanding to modeling the growth of a population behaving with specified constraints.

This Topic Fit?

In grade 8, students develop an understanding of what a function is as well as what makes a function linear as they compare and contrast linear and non-linear functions. The work in bivariate data also relies upon work with linear functions, as students draw a line of best fit for data, write an equation to represent that line and then use their equation to make predictions. In the study of systems of linear equations, students rely upon their ability to write and graph equations to find the solution to a system of equations graphically. A strong foundation in the understanding of linear functions and the interrelation between various representations of functions is essential to the understanding of more complex functions, such as rational functions. The square root symbol is introduced in grade 8 and square roots are used to solve the quadratic equations that arise when using the Pythagorean Theorem.

In Math I, students further develop their understanding of functions and learn to use function notation. The emphasis in Math I is on comparing and contrasting linear functions with exponential functions, recognizing the additive growth of the former and the multiplicative growth of the latter. Math I students continue to work with bivariate data and the fitting of equations to functions that are represented as graphs, tables of values and verbal descriptions. The end behavior of exponential functions is explored and the idea of an asymptote is first presented in Math I. These will be further developed and expanded into other types of asymptotes in Math III.

In Math II, the variety of functions with which students have experience greatly increases. Quadratic, absolute value, step, and piecewise defined functions are all introduced. Key features are used to identify which function best matches a given situation or graph. These same key features are investigated in Math III for rational functions.

Coherence

Connections to other Math III/Algebra II Topics

Throughout high school, a significant part of the study of mathematics is the study of functions. In Math III or Algebra II, several new function families are introduced. One of the early units of the year introduces polynomial functions beyond linear and quadratic. The units that follow present radical and rational functions and also teach how to solve radical and rational equations. In this unit, exponential functions will be revisited and studied along with their inverses, the logarithmic functions. In the same pattern as previous units, strategies are taught for solving the equations that arise when working with exponential functions. Later in the year, sinusoidal functions will be studied. As Math III/Algebra II finishes, students will have a wide variety of functions available to them to model many different real world situations.

Essential Questions

Essential Question # 1
What is a geometric sequence? Geometric series?

Essential Question # 2
What is an exponential function?

What are its characteristsics compared to other functions?

Essential Question # 3
What is a logarithmic function?

What are its characteristsics compared to other functions?

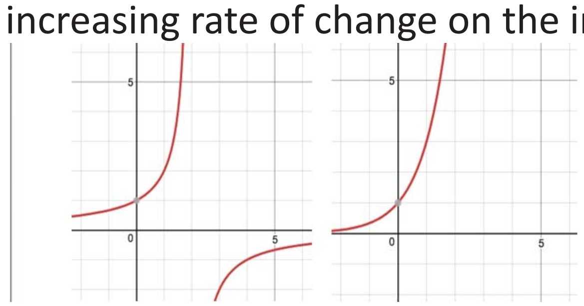
Common Misconceptions

All functions that use exponents are exponential.

Exponential functions have specific shared key features. One of these features is a common ratio of outputs for a fixed interval of inputs. Another feature is a graph with one end behavior that approaches a horizontal asymptote while the other end behavior approaches either infinity or negative infinity. The domain is unrestricted while the range is limited to one side of the asymptote. These key features distinguish exponential functions from other functions that have numerical rather than variable expressions in the exponent, such as x^2 or x^3. It is not uncommon for students to look at a quadratic function (in any form: equation, data table, or graph) and say that it is "exponential." This is wrong and more than just than just a semantic difference. The idea and name "exponential" should be connected to the key features that define this family of functions.

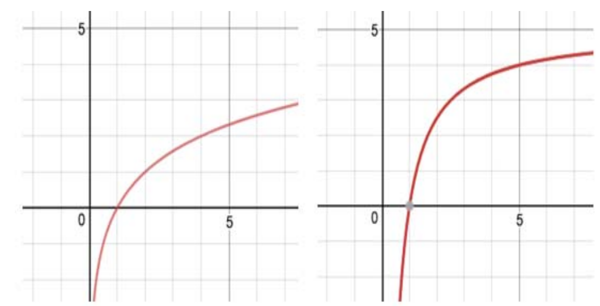
Exponential functions have vertical asymptotes and logarithmic functions have horizontal asymptotes.

Once students have studied rational functions and been exposed to vertical asymptotes they often begin to imagine they see them in places where they do not exist. The first image below is a portion of the graph of the function $f(x)=-2/(x-2)$. The second image is a portion of the graph of the function $g(x)=3^x$. Both graphs are increasing with an increasing rate of change on the interval $(-\infty,2)$. Both



graphs approach a horizontal asymptote at $y=0$ as x approaches negative infinity. In many ways, the upper part of the rational function appears very similar to the exponential function. We cannot rely on the image, however, to decide if there is a vertical asymptote. For the exponential function, we know that the function is defined continuously as it approaches 3 and that $3^2=9$. Therefore, the exponential function cannot have a vertical asymptote at x=2. In fact, the exponential function is defined continuously for all real number inputs which rules out the possibility of a vertical asymptote anywhere.

A similar visual miscue can lead to believing that logarithmic graphs have horizontal asymptotes. Consider the two graphs below defined by the expressions $\log_2 x$ and $-5/x+5$. It would be easy



to mistake one for the other. In this case, it requires more sophisticated thinking to assert that it is possible to find a sufficiently large input for the logarithm function so that the value of its output exceeds any given value. Said another way: the values of the logarithm function increase without bound.

Roots are the inverses needed to solve exponential equations

The operation of taking the logarithm of a number can be difficult to accept as an operation by many students. It is the first new operation to be introduced to students since roots in middle school. In addition, instead of having a symbol as other operations have, it uses the first three letters from its full name and also uses a subscript. As a result, it probably should not be surprising that some students do not readily adapt to using logarithms to solve exponential equations. This is especially true when the base of the exponential equation is a familiar number, such as 2 or 3. Watch for students who attempt to use a square root to solve equations such as $2^x=1000$.

Academic Language

- exponential
- logarithmic
- sequence
- series
- inverse
- recursive
- explicit

Real-World Application

- Population growth
- Investment returns
- Rates of cooling