SOLUTIONS

MOCK MIDTERM

PEYAM RYAN TABRIZIAN

Instructions: This is a mock midterm, designed to give you some practice for the actual midterm. It will be similar in length and in spirit to the actual midterm, but do **NOT** expect the questions on the midterm to be the same; some will be easier, some will be harder. So please also look at the study guide and the suggested homework for a more complete study experience!

1		15
2		20
3		15
4		20
5		15
6		15
Total		100

Date: Friday, May 4, 2018.

1. *(15 points)* Find the equation of the tangent line of $r = 1 + 2\cos(\theta)$ at $\theta = \frac{\pi}{3}$.

SLOPE 1) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(r\sin(\theta))'}{(r\cos(\theta))'}$

$= \frac{r'\sin(\theta) + r\cos(\theta)}{r'\cos(\theta) - r\sin(\theta)}$

$= \frac{(-2\sin(\theta))\sin(\theta) + (1+2\cos(\theta))\cos(\theta)}{(-2\sin(\theta))\cos(\theta) - (1+2\cos(\theta))\sin(\theta)}$

AT $\theta = \pi/3$, $\cos(\pi/3) = \frac{1}{2}$, $\sin(\pi/3) = \frac{\sqrt{3}}{2}$, so

$\frac{dy}{dx} = \frac{(-2)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(1 + 2\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)}{(-2)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) - \left(1 + 2\left(\frac{1}{2}\right)\right)\left(\frac{\sqrt{3}}{2}\right)}$

$= \frac{-\frac{3}{2} + 1}{-\frac{\sqrt{3}}{2} - \sqrt{3}} = \frac{-\frac{1}{2}}{-\frac{3\sqrt{3}}{2}} = +\frac{1}{3\sqrt{3}}$

POINT 2) AT $\theta = \pi/3$, $r = 1 + 2\cos\left(\frac{\pi}{3}\right) = 1 + 2\left(\frac{1}{2}\right) = 2$

THEN $x = r\cos(\theta) = 2\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$

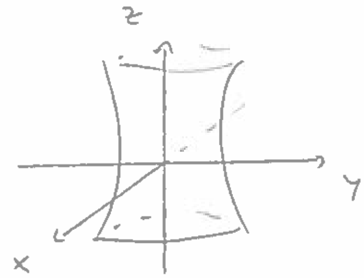
$y = r\sin(\theta) = 2\sin\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$ $(1, \sqrt{3})$

EQUATION 3) $\boxed{y - \sqrt{3} = \left(+\frac{1}{3\sqrt{3}}\right)(x - 1)}$

2. *(20 points, 4 points each)* For each of the following surfaces, put the name and draw a small sketch of the figure. It's ok if it's not drawn to scale, but the direction needs to be correct.

(a) $x^2 + 2y^2 - 3z^2 = 4$

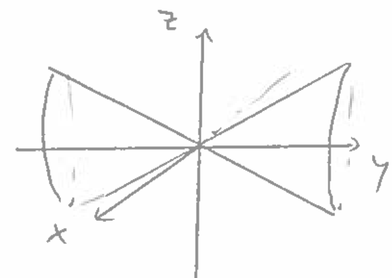
HYPERBOLOID OF ONE SHEET / DRESS



(b) $x^2 - y^2 + z^2 = 0$

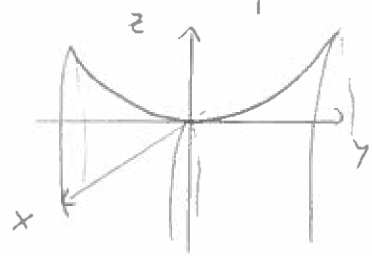
$\implies y^2 = x^2 + z^2$

CONE (IN Y-DIRECTION)



(c) $z = 3y^2 - 5x^2$

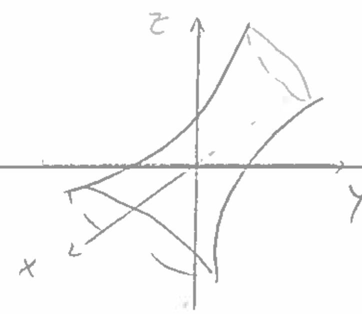
HYPERBOLIC PARABOLOID / SADDLE



(d) $2x^2 - 5y^2 - 6z^2 = -2$

$\implies -x^2 + \frac{5}{2}y^2 + 3z^2 = 1$

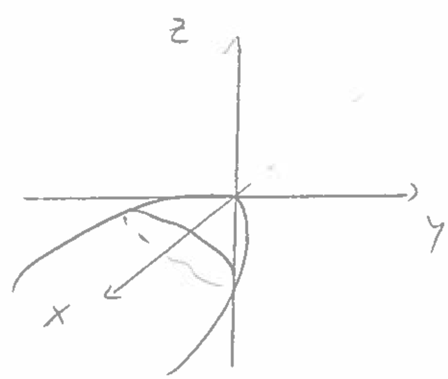
HYPERBOLOID OF ONE SHEET / DRESS
IN X-DIRECTION



(e) $x = y^2 + z^2$

(ELLIPTIC) PARABOLOID

IN THE X-DIRECTION



3. *(15 points)* Find the equation of the tangent line to the curve

$$\mathbf{r}(t) = \langle e^{-t}\cos(t), e^{-t}\sin(t), e^{-t}\rangle$$

at the point $(1, 0, 1)$.

1) FIND t $\qquad \langle e^{-t}\cos(t), e^{-t}\sin(t), e^{-t}\rangle = (1,0,1)$

$$\Rightarrow \begin{cases} e^{-t}\cos(t) = 1 \rightsquigarrow e^{-0}\cos(0) = 1 \checkmark \\ e^{-t}\sin(t) = 0 \rightsquigarrow e^{-0}\sin(0) = 0 \checkmark \\ e^{-t} = 1 \rightsquigarrow -t = \ln(1) = 0 \Rightarrow \underline{t=0} \end{cases}$$

So $\underline{t = 0}$

2) POINT $(1, 0, 1)$

SLOPE $\quad r'(t) = \langle -e^{-t}\cos(t) - e^{-t}\sin(t), -e^{-t}\sin(t) + e^{-t}\cos(t), -e^{-t}\rangle$

$\underline{t=0}$ $\quad r'(0) = \langle -(1)(1) - (1)(0), -(1)(0) + (1)(1), -1\rangle$

$= \langle -1, 1, -1\rangle$

EQUATION
$$\begin{cases} x(t) = 1 - t \\ y(t) = t \\ z(t) = 1 - t \end{cases}$$

$(1,0,1)$ $\langle -1, 1, -1\rangle$

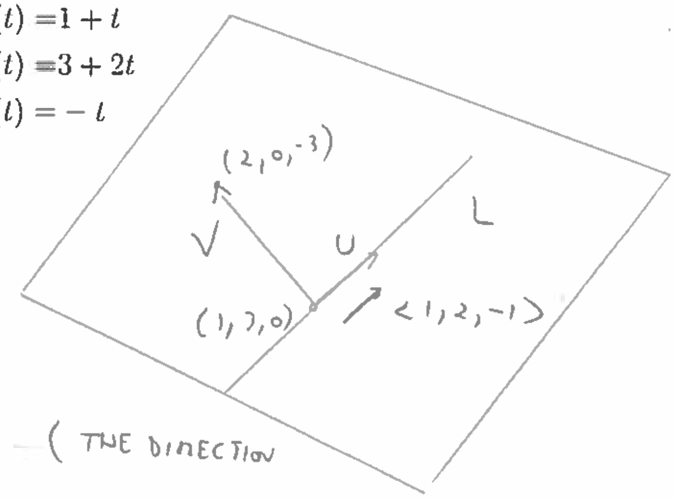
4. *(20 points, 10 points each)* **Note:** Here I put 2 sub-parts just to give you more practice; a more reasonable exam question like that would only have one sub-part.

(a) Find the equation of the plane containing the point $(2, 0, -3)$ and the line L with equation:

$$x(t) = 1 + t$$
$$y(t) = 3 + 2t$$
$$z(t) = -t$$

1) POINT $(2,0,-3)$

2) NORMAL VECTOR



FIRST OF ALL, $U = \langle 1, 2, -1\rangle$ (THE DIRECTION VECTOR TO L) IS ON THE PLANE, SO ONLY NEED TO FIND ONE MORE VECTOR.

BUT SINCE $(2,0,-3)$ AND $\underbrace{(1+0,\ 3+2(0),\ -(0))}_{\text{SETTING } t=0 \text{ IN } L} = (1,3,0)$

ARE ON THE PLANE, SO IS THE VECTOR

$V = \underbrace{\langle 1-2,\ 3-0,\ 0-(-3)\rangle}_{(1,3,0)-(2,0,-3)} = \langle -1, 3, 3\rangle$ (THE VECTOR JOINING THE POINTS)

HENCE $N = U \times V = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -1 & 3 & 3 \end{vmatrix} = i\,(6+3) - j\,(3-1) + k\,(3+2)$

$N = \langle 9, -2, 5\rangle$

$N = \langle 9,-2,5\rangle$
POINT $(2,0,-3)$

3) EQUATION $9(x-2) - 2(y-0) + 5(z-(-3)) = 0$

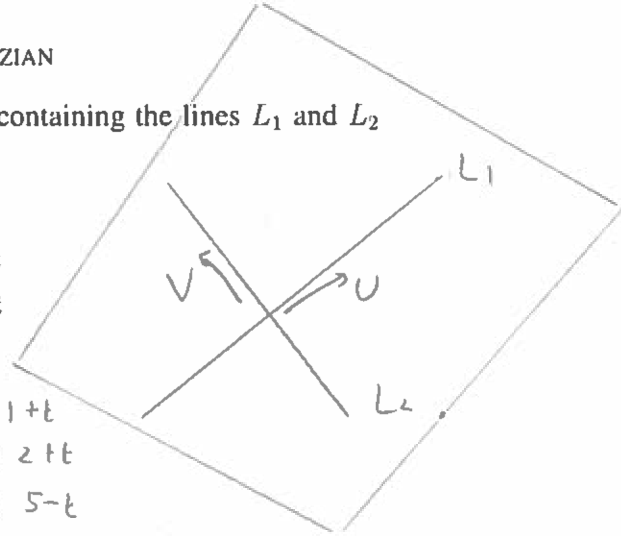
$$\boxed{9(x-2) - 2y + 5(z+3) = 0}$$

(b) Find the equation of the plane containing the lines L_1 and L_2 with equations

$$x(t) = 2 - t$$
$$y(t) = 3 + 2t$$
$$z(t) = 4 - 3t$$

and

$$x(t) = 1 + t$$
$$y(t) = 2 + t$$
$$z(t) = 5 - t$$



1) NORMAL VECTOR

THE DIRECTION VECTORS $U = \langle -1, 2, -3 \rangle$ OF L_1 AND $V = \langle 1, 1, -1 \rangle$ OF L_2

ARE BOTH IN THE PLANE, SO

$$N = U \times V = \begin{vmatrix} i & j & k \\ -1 & 2 & -3 \\ 1 & 1 & -1 \end{vmatrix} = i(-2+3) - j(1+3) + k(-1-2)$$

$$= \langle 1, -4, -3 \rangle$$

2) POINT

JUST CHOOSE ONE POINT ON L_1 OR L_2

$t = 0$ IN L_1 GIVES $(2-0, 3+2(0), 4-3(0)) = (2, 3, 4)$

3) EQUATION $N = \langle 1, -4, -3 \rangle$, POINT $= (2, 3, 4)$

$$1(x-2) - 4(y-3) - 3(z-4) = 0$$

$$\boxed{(x-2) - 4(y-3) - 3(z-4) = 0}$$

5. *(15 points)* Use polar coordinates to find the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{e^{-(x^2+y^2)}-1}{x^2+y^2}$$

$x = r\cos(\theta)$, $y = r\sin(\theta)$

$x^2+y^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$

So $\lim = \lim_{r\to 0} \frac{e^{-r^2}-1}{r^2}$ $\quad \frac{0}{0}$

$\overset{\hat{H}}{=} \lim_{r\to 0} \frac{(-\cancel{2r})e^{-r^2}}{\cancel{2r}}$ $\quad (\hat{H} = \text{L'Hôpital's rule})$

$= -e^{-(0^2)}$

$= -1$

6. *(15 points)* Find the equation of the tangent plane to the function $z = \sin(xy)$ at the point $(2\sqrt{\pi}, \frac{1}{2}\sqrt{\pi})$

$$f(x,y) = \sin(xy)$$

$\cdot)$ $$f_x(x,y) = \cos(xy)\,(y)$$

$$f_x\left(2\sqrt{\pi}, \tfrac{1}{2}\sqrt{\pi}\right) = \cos\left(\cancel{2}\sqrt{\pi}\ \tfrac{1}{\cancel{2}}\sqrt{\pi}\right)\left(\tfrac{1}{2}\sqrt{\pi}\right)$$

$$= \cos(\pi)\,\frac{\sqrt{\pi}}{2} = -\frac{\sqrt{\pi}}{2}$$

$\cdot)$ $$f_y(x,y) = \cos(xy)\,(x)$$

$$f_y\left(2\sqrt{\pi}, \tfrac{1}{2}\sqrt{\pi}\right) = \cos\left(\cancel{2}\sqrt{\pi}\ \tfrac{1}{\cancel{2}}\sqrt{\pi}\right)\left(2\sqrt{\pi}\right)$$

$$= \cos(\pi)\,2\sqrt{\pi} = -2\sqrt{\pi}$$

$\cdot)$ $$f\left(2\sqrt{\pi}, \tfrac{1}{2}\sqrt{\pi}\right) = \sin\left(\cancel{2}\sqrt{\pi}\ \tfrac{1}{\cancel{2}}\sqrt{\pi}\right) = \sin(\pi) = 0$$

$\cdot)$ SO EQUATION IS

$$\boxed{z - 0 = -\frac{\sqrt{\pi}}{2}\left(x - 2\sqrt{\pi}\right) - 2\sqrt{\pi}\left(y - \frac{\sqrt{\pi}}{2}\right)}$$