

MATH 2E REVIEW FOR FINAL

The final is in the usual classroom, Wed, December 12, 1:30pm – 3:30pm, 8–9 problems, covering Chapter 15 and 16 of Stewart calculus, no notes.

Chapter 15.

- (1) Calculate $\iint_R ye^{xy}dA$, where $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3\}$.
- (2) Calculate $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dx dy$.
- (3) Calculate $\iiint_E z dV$, where E is bounded by the planes $y = 0$, $z = 0$, $x + y = 2$ and the cylinder $y^2 + z^2 = 1$ in the first octant.
- (4) Calculate $\iiint_E yz dV$ where E lies above the plane $z = 0$, below the plane $z = y$, and inside the cylinder $x^2 + y^2 = 4$.
- (5) Calculate $\iiint_H z^3 \sqrt{x^2 + y^2 + z^2} dV$, where H is the solid hemisphere that lies above the xy -plane and has center the origin and radius 1.
- (6) Evaluate $\iint_R \frac{x-y}{x+y} dA$ where R is the square with vertices $(0, 2)$, $(1, 1)$, $(2, 2)$ and $(1, 3)$.
- (7) Find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes. Consider the transformation $x = u^2$, $y = v^2$, and $z = w^2$.
- (8) Evaluate $\iint_R xy dA$, where R is the square with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$.
- (9) Given a curve $r(t) = \langle 1 + t, t^2, t^3 \rangle$, find the area of the triangle with vertices $r(-1)$, $r(1)$ and $r(0)$.

Chapter 16.

- (1) Evaluate $\int_C x ds$, where C is the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.
- (2) Evaluate $\int_C y dx + (x + y^2) dy$, C is the ellipse $4x^2 + 9y^2 = 36$ with counter clockwise orientation.
- (3) Evaluate $\int_C F \cdot dr$, where $F = \langle \sqrt{xy}, e^y, xz \rangle$, C is given by $r(t) = \langle t^4, t^2, t^3 \rangle$, $0 \leq t \leq 1$.
- (4) Compute $\text{curl } F$ where $F = \langle e^y, xe^y + e^z, ye^z \rangle$. Then compute the line integral $\int_C F \cdot dr$ where C is **any** curve from $(0, 2, 0)$ to $(4, 0, 3)$. Hint: fundamental theorem of line integrals.
- (5) Verify Green's theorem is true for the line integral $\int_C xy^2 dx - x^2 y dy$, where C consists of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-1, 1)$.
- (6) Find the area of the part of the surface $z = x^2 + 2y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$.
- (7) Find an equation of the tangent plane at the point $(4, -2, 1)$ to the parametric surface S given by $r(u, v) = \langle v^2, -uv, u^2 \rangle$, $0 \leq u \leq 3$, $-3 \leq v \leq 3$.

- (8) Evaluate $\iint_S z dS$ and $\iint_S x dS$ where S is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$.
- (9) Evaluate $\iint_S x^2 z + y^2 z dS$, where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$.
- (10) Evaluate $\iint_S F \cdot dS$ where $F = \langle xz, -2y, 3x \rangle$ and S is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation.
- (11) Verify Stokes' theorem is true for $F = \langle x^2, y^2, z^2 \rangle$, where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy -plane and S has upward orientation.
- (12) Evaluate $\int_C F \cdot dr$ where $F = \langle xy, yz, zx \rangle$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, oriented counter clockwise as viewed from above.
- (13) Calculate $\iint_S F \cdot dS$ where $F = \langle x^3, y^3, z^3 \rangle$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.
- (14) Compute the outward flux of $F = \left\langle \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle$ through the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.
- (15) Compute $\int_C F \cdot dr$ where $F = \left\langle \frac{2x^3+2xy^2-2y}{x^2+y^2}, \frac{2y^3+2x^2y+2x}{x^2+y^2} \right\rangle$ around any simple closed curve containing the origin $(0, 0)$.
- (16) Find the positively oriented simple closed curve C for which the value of the line integral $\int_C (y^3 - y)dx - 2x^3dy$ is a maximum.

Select Solutions.

15.1

$$\begin{aligned}
 \int_0^3 \int_0^2 ye^{xy} dx dy &= \int_0^3 (e^{xy} \Big|_0^2) dy \\
 &= \int_0^3 e^{2y} - 1 dy \\
 &= \frac{1}{2} e^{2y} \Big|_0^3 - 3 \\
 &= \frac{1}{2} e^6 - \frac{7}{2}.
 \end{aligned}$$

15.2

$$\begin{aligned}
\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dx dy &= \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3} dy dx \\
&= \frac{1}{2} \int_0^1 y^2 \Big|_0^{x^2} \frac{e^{x^2}}{x^3} dx \\
&= \frac{1}{2} \int_0^1 x e^{x^2} dx \\
&= \frac{1}{4} e^{x^2} \Big|_0^1 \\
&= \frac{1}{4} (e - 1).
\end{aligned}$$

15.3

Let D be a quarter of the unit circle in first quadrant of yz plane. Then

$$\iint_D \int_0^{2-y} z dx dA = \iint_D z(2-y) dA$$

Let $y = r \cos(\theta)$, $z = r \sin(\theta)$, then $dA = r dr d\theta$ so that

$$\begin{aligned}
2 \int_0^{\pi/2} \int_0^1 r^2 \sin(\theta) dr d\theta - \int_0^{\pi/2} \int_0^1 r^3 \sin(\theta) \cos(\theta) dr d\theta &= \frac{2}{3} - \frac{1}{8} \\
&= \frac{13}{24}.
\end{aligned}$$

15.4

Let D be the upper half of the disk of radius 2 on the xy plane.

$$\begin{aligned}
\iint_D \int_0^y yz dz dA &= \frac{1}{2} \iint_D yz^2 \Big|_0^y dA \\
&= \frac{1}{2} \iint_D y^3 dA \\
&= \frac{1}{2} \int_0^2 \int_0^\pi r^4 \sin^3(\theta) d\theta dr \\
&= \frac{1}{5} (2^4) \frac{4}{3} = \frac{64}{15}
\end{aligned}$$

15.8

The function is odd across the symmetric domain, hence the integral is zero.

If you want to, you could have also done: Let $u = y - x$ and $v = y + x$. Then $x = \frac{1}{2}(v - u)$ and $y = \frac{1}{2}(v + u)$. Then $dx dy = \frac{1}{2} du dv$ so

$$\begin{aligned}
\iint_R xy dA &= \frac{1}{2} \int_0^2 \int_{-2}^0 \frac{v^2 - u^2}{4} du dv \\
&= \frac{1}{8} \int_0^2 (2v^2 - \frac{8}{3}) dv = 0
\end{aligned}$$

16.1

The parametrization is given by $r(t) = (t, t^2)$, $0 \leq t \leq 1$. Then $r'(t) = (1, 2t)$ so $ds = \|r'(t)\| = \sqrt{1 + 4t^2}$.

$$\int_0^1 t\sqrt{1 + 4t^2} dt = \frac{1}{12}(5\sqrt{5} - 1).$$

16.2

By Green's theorem,

$$\int_C y dx + (x + y^2) dy = \iint_D 0 dA = 0.$$

16.3

After parametrizing, we get

$$\int_0^1 4t^6 + 2te^{t^2} + 3t^9 dt = e - \frac{9}{70}$$

16.4

$\text{curl } F = 0$, therefore, there is an f such that $F = \nabla f$. Doing the usual steps we find that $f(x, y, z) = xe^y + ye^z$ hence

$$\int_C F \cdot dr = f(4, 0, 3) - f(0, 2, 0) = 2.$$

16.6

Using the parametrization $r(x, y) = (x, y, x^2 + 2y)$, we get $\|r_x \times r_y\| = \sqrt{5 + 4x^2}$. So

$$\int_0^1 \int_0^{2x} \sqrt{5 + 4x^2} dy dx = \int_0^1 2x\sqrt{5 + 4x^2} dx = \frac{1}{6}(27 - 5\sqrt{5}).$$

16.7

The tangent vectors are $r_u = \langle 0, -v, 2u \rangle$ and $r_v = \langle 2v, -u, 0 \rangle$, the normal vector is $r_u \times r_v = \langle 2u^2, 4uv, 2v^2 \rangle$. Since $u^2 = 1$ and $u \geq 0$, we must have $u = 1$. If $u = 1$ the $-v = -2$ so that $v = 2$. So it is at the point $u = 1, v = 2$. Plugging this in, we get $n = \langle 2, 8, 8 \rangle$, hence the equation is given by $2(x - 4) + 8(y + 2) + 8(z - 1) = 0$.

16.9

Under the parametrization $r(x, y) = (x, y, 4 + x + y)$, we have $r_x \times r_y = \langle -1, -1, 1 \rangle$ so using polar coordinates,

$$\iint_{x^2 + y^2 \leq 4} (x^2 + y^2)(4 + x + y)\sqrt{3} dA = 32\pi\sqrt{3}$$

16.11

We want to show $\int_{\partial S} F \cdot dr = \iint_S \text{curl } F \cdot dS$. The boundary is a circle on the x, y plane so that $r(t) = \langle \cos(t), \sin(t), 0 \rangle$. Then

$$\int_{\partial S} F \cdot dr = \int_0^{2\pi} (-\cos^2(t)\sin(t) + \sin^2(t)\cos(t)) dt = 0.$$

By direct computation, we have $\text{curl } F = \langle 0, 0, 0 \rangle$.

16.15

There is a typo in the original question. It is fixed in this version. One computes $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ so that according to Green's theorem, the line integral would be zero, however, this is not true because the vector field is not differentiable at $(0, 0)$. It is not even defined there. So we consider the region bounded by a unit circle and some arbitrary closed curve. By reversing orientation, this

region would enclose a region that does not contain the origin so that Green's theorem can be applied. In conclusion, we get

$$\int_C F \cdot dr = \int_{x^2+y^2=1} F \cdot dr.$$

On the unit circle, one computes that $\int_C F \cdot dr = 4\pi$.

16.16

By Green's theorem,

$$\int_C (y^3 - y)dx - 2x^3dy = \iint_D 1 - 6x^2 - 3y^2 dA.$$

The integral is maximum if we integrate over the region with $f \geq 0$ for $\iint_D f dA$. Hence the domain D should be given by $1 \geq 6x^2 + 3y^2$ and so the boundary is $1 = 6x^2 + 3y^2$.