MATH 2E REVIEW FOR FINAL

The final is in the usual classroom, Wed, December 12, 1:30pm – 3:30pm, 8–9 problems, covering Chapter 15 and 16 of Stewart calculus, no notes.

Chapter 15.

(1) Calculate \( \int \int_R ye^{xy} \, dA \), where \( R = \{ (x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3 \} \).

(2) Calculate \( \int_0^1 \int_0^1 ye^{x^2} \, dx \, dy \).

(3) Calculate \( \int \int_E zdV \), where \( E \) is bounded by the planes \( y = 0, z = 0, x + y = 2 \) and the cylinder \( y^2 + z^2 = 1 \) in the first octant.

(4) Calculate \( \int \int_E yzdV \) where \( E \) lies above the plane \( z = 0 \), below the plane \( z = y \), and inside the cylinder \( x^2 + y^2 = 4 \).

(5) Calculate \( \int \int_H z^3 \sqrt{x^2 + y^2 + z^2} \, dV \), where \( H \) is the solid hemisphere that lies above the \( xy \)-plane and has center the origin and radius 1.

(6) Evaluate \( \int \int_R x - \frac{y}{x + y} \, dA \) where \( R \) is the square with vertices \((0, 2), (1, 1), (2, 2)\) and \((1, 3)\).

(7) Find the volume of the region bounded by the surface \( \sqrt{x} + \sqrt{y} + \sqrt{z} = 1 \) and the coordinate planes. Consider the transformation \( x = u^2, y = v^2, \) and \( z = w^2 \).

(8) Evaluate \( \int \int_R xy \, dA \), where \( R \) is the square with vertices \((0, 0), (1, 1), (2, 0), \) and \((1, -1)\).

(9) Given a curve \( r(t) = \langle 1 + t, t^2, t^3 \rangle \), find the area of the triangle with vertices \( r(-1), r(1) \) and \( r(0) \).

Chapter 16.

(1) Evaluate \( \int_C x \, ds \), where \( C \) is the arc of the parabola \( y = x^2 \) from \((0, 0)\) to \((1, 1)\).

(2) Evaluate \( \int_C y \, dx + (x + y^2) \, dy \), \( C \) is the ellipse \( 4x^2 + 9y^2 = 36 \) with counter clockwise orientation.

(3) Evaluate \( \int_C F \cdot dr \), where \( F = \langle \sqrt{xy}, ey, xz \rangle \), \( C \) is given by \( r(t) = \langle t^4, t^2, t^3 \rangle, 0 \leq t \leq 1 \).

(4) Compute \( \text{curl} \, F \) where \( F = \langle e^y, xe^y + e^z, ye^z \rangle \). Then compute the line integral \( \int_C F \cdot dr \) where \( C \) is any curve from \((0, 2, 0)\) to \((4, 0, 3)\). Hint: fundamental theorem of line integrals.

(5) Verify Green’s theorem is true for the line integral \( \int_C xy^2 \, dx - x^2y \, dy \), where \( C \) consists of the parabola \( y = x^2 \) from \((-1, 1)\) to \((1, 1)\) and the line segment from \((1, 1)\) to \((-1, 1)\).

(6) Find the area of the part of the surface \( z = x^2 + 2y \) that lies above the triangle with vertices \((0, 0), (1, 0)\) and \((1, 2)\).

(7) Find an equation of the tangent plane at the point \((4, -2, 1)\) to the parametric surface \( S \) given by \( r(u, v) = \langle v^2, -uv, u^2 \rangle, 0 \leq u \leq 3, -3 \leq v \leq 3 \).
(8) Evaluate $\int \int_S zdS$ and $\int \int_S xdS$ where $S$ is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$.
(9) Evaluate $\int \int_S x^2 + y^2 z dS$, where $S$ is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$.
(10) Evaluate $\int \int_S F \cdot dS$ where $F = \langle xz, -2y, 3x \rangle$ and $S$ is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation.
(11) Verify Stokes’ theorem is true for $F = \langle x^2, y^2, z^2 \rangle$, where $S$ is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the $xy$-plane and $S$ has upward orientation.
(12) Evaluate $\int_C F \cdot dr$ where $F = \langle xy, yz, zx \rangle$ and $C$ is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, oriented counter clockwise as viewed from above.
(13) Calculate $\int \int_S F \cdot dS$ where $F = \langle x^3, y^3, z^3 \rangle$ and $S$ is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.
(14) Compute the outward flux of $F = \left( \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right)$ through the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.
(15) Compute $\int_C F \cdot dr$ where $F = \left( \frac{2x^3+2xy^2-2x}{x^2+y^2}, \frac{2y^3+2x^2y+2y}{x^2+y^2} \right)$ around any simple closed curve containing the origin $(0, 0)$.
(16) Find the positively oriented simple closed curve $C$ for which the value of the line integral $\int_C (y^3 - y) dx - 2x^3 dy$ is a maximum.

Select Solutions.
15.1

$$\int_0^3 \int_0^2 ye^{xy} dy dx = \int_0^3 \left( e^{xy} \right)^2|_0^2 dy$$
$$= \int_0^3 e^{2y} - 1 dy$$
$$= \frac{1}{2} e^{2y} \bigg|_0^3$$
$$= \frac{1}{2} e^{6} - \frac{7}{2}.$$
15.2
\[
\int_0^1 \int_0^1 \frac{ye^x}{x^3} \, dx \, dy = \int_0^1 \frac{ye^x}{x^3} \, dy \, dx \\
= \frac{1}{2} \int_0^1 y^2 e^x \, dx \\
= \frac{1}{2} \int_0^1 xe^x \, dx \\
= \frac{1}{4} e^x \bigg|_0^1 \\
= \frac{1}{4} (e - 1).
\]

15.3
Let \( D \) be a quarter of the unit circle in first quadrant of \( yz \) plane. Then
\[
\int \int_D \int_0^{2-y} zdxA = \int \int_D z(2 - y)dA
\]
Let \( y = r \cos(\theta), z = r \sin(\theta), \) then \( dA = r dr d\theta \) so that
\[
2 \int_0^{\pi/2} \int_0^1 r^2 \sin(\theta) dr d\theta - \int_0^{\pi/2} \int_0^1 r^3 \sin(\theta)\cos(\theta) dr d\theta = \frac{2}{3} - \frac{1}{8} = \frac{13}{24}.
\]

15.4
Let \( D \) be the upper half of the disk of radius 2 on the \( xy \) plane.
\[
\int \int_D \int_0^y yzdA = \frac{1}{2} \int \int_D yz^2 \bigg|_0^y \, dA \\
= \frac{1}{2} \int \int_D y^3 dA \\
= \frac{1}{2} \int_0^2 \int_0^\pi r^4 \sin^3(\theta) d\theta dr \\
= \frac{1}{5} (2^4) \frac{4}{3} = \frac{64}{15}
\]

15.8
The function is odd across the symmetric domain, hence the integral is zero.
If you want to, you could have also done: Let \( u = y - x \) and \( v = y + x. \) Then \( x = \frac{1}{2}(v - u) \) and \( y = \frac{1}{2}(v + u). \) Then \( dx dy = \frac{1}{2} dudv \) so
\[
\int \int_R xydA = \frac{1}{2} \int_0^2 \int_{-2}^{2} \frac{v^2 - u^2}{4} \\
= \frac{1}{8} \int_0^2 (2v^2 - \frac{8}{3}) dv = 0
\]

16.1
The parametrization is given by \( r(t) = (t, t^2), \) \( 0 \leq t \leq 1. \) Then \( r'(t) = (1, 2t) \) so \( ds = \|r'(t)\| = \sqrt{1 + 4t^2}. \)

\[
\int_0^1 t\sqrt{1 + 4t^2} dt = \frac{1}{12}(5\sqrt{5} - 1).
\]

16.2
By Green’s theorem,

\[
\int_C ydx + (x + y^2)dy = \iint_D 0dA = 0.
\]

16.3
After parametrizing, we get

\[
\int_0^1 4t^6 + 2te^{t^2} + 3t^9 dt = e - \frac{9}{70}.
\]

16.4
\( \text{curl } F = 0, \) therefore, there is an \( f \) such that \( F = \nabla f. \) Doing the usual steps we find that

\[
\int_C F \cdot n = f(4, 0, 3) - f(0, 2, 0) = 2.
\]

16.6
Using the parametrization \( r(x, y) = (x, y, x^2 + 2y), \) we get \( \|r_x \times r_y\| = \sqrt{5 + 4x^2}. \) So

\[
\int_0^1 \int_0^{2x} \sqrt{5 + 4x^2} dy dx = \int_0^1 2x\sqrt{5 + 4x^2} dx = \frac{1}{6}(27 - 5\sqrt{5}).
\]

16.7
The tangent vectors are \( r_u = (0, -v, 2u) \) and \( r_v = (2v, -u, 0), \) the normal vector is \( r_u \times r_v = (2u^2, 4uv, 2v^2). \) Since \( u^2 = 1 \) and \( u \geq 0, \) we must have \( u = 1. \) If \( u = 1 \) the \( -v = -2 \) so that \( v = 2. \) So it is at the point \( u = 1, v = 2. \) Plugging this in, we get \( n = (2, 8, 8), \) hence the equation is given by \( 2(x - 4) + 8(y + 2) + 8(z - 1) = 0. \)

16.9
Under the parametrization \( r(x, y) = (x, y, 4 + x + y), \) we have \( r_x \times r_y = (-1, -1, 1) \) so using polar coordinates,

\[
\iint_{x^2+y^2 \leq 4} (x^2 + y^2)(4 + x + y)\sqrt{3} dA = 32\pi\sqrt{3}.
\]

16.11
We want to show \( \int_{\partial S} F \cdot dr = \iint_S \text{curl } F \cdot dS. \) The boundary is a circle on the \( x, y \) plane so that \( r(t) = (\cos(t), \sin(t), 0). \) Then

\[
\int_{\partial S} F \cdot dr = \int_0^{2\pi} (-\cos^2(t)\sin(t) + \sin^2(t)\cos(t)) dt = 0.
\]

By direct computation, we have \( \text{curl } F = (0, 0, 0). \)

16.15
There is a typo in the original question. It is fixed in this version. One computes \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \) so that according to Green’s theorem, the line integral would be zero, however, this is not true because the vector field is not differentiable at \( (0, 0). \) It is not even defined there. So we consider the region bounded by a unit circle and some arbitrary closed curve. By reversing orientation, this
region would enclose a region that does not contain the origin so that Green’s theorem can be applied. In conclusion, we get

\[ \int_C \mathbf{F} \cdot dr = \int_{x^2+y^2=1} \mathbf{F} \cdot dr. \]

On the unit circle, one computes that \( \int_C \mathbf{F} \cdot dr = 4\pi \).

16.16

By Green’s theorem,

\[ \int_C (y^3 - y)dx - 2x^3 dy = \iint_D 1 - 6x^2 - 3y^2 dA. \]

The integral is maximum if we integrate over the region with \( f \geq 0 \) for \( \iint_D f dA \). Hence the domain \( D \) should be given by \( 1 \geq 6x^2 + 3y^2 \) and so the boundary is \( 1 = 6x^2 + 3y^2 \).