# MATH 54 – HINTS – BANK

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This document contains hints to the following Math 54 problems I've compiled in the past. The problems in **bold** have complete solutions:

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## 1. Section 1.1: Systems of linear equations

**1.1.15.** All you have to do are row-reductions until it is easier to see whether the equation has a solution or not. In particular, if one of the rows is of the form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & b \end{bmatrix}$$

then the system has no solution!

**1.1.20.** Solve the system as if $h$ was a number! It might be useful to divide the second row by $-2$. Again, use the fact that if one of the rows is of the form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & b \end{bmatrix}$$

then the system has no solution!

The answer is $\boxed{h \neq -4}$

**1.1.28.** One way to do this is to start with the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

And, for example, multiply each row by 2, or add the third row to the second, or interchange the first and second row. There's a whole world of different possibilities; just make sure not to destroy the system by either making it inconsistent, or by adding infinite solutions.

## Section 1.2: Row Reduction and Echelon Forms

**1.2.5.** Just argue by the number of pivots! There are three possible echelon forms here:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \circ & \star \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \circ & \star \\ 0 & \circ \end{bmatrix}$$

Where $\circ$ stands for 'pivot,' and $\star$ is any number (could be zero or not).

**1.2.11.** Row-reduce the matrix (we divided the second row by 3, the third row by 2; and then we subtracted the first row from the second row and the third row):

$$\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -2 & 4 & 0 \\ 3 & -2 & 4 & 0 \\ 3 & -2 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In terms of equations, this becomes $3x_1 - 2x_2 + 4x_3 = 0$, that is $3x_1 = 2x_2 - 4x_3$, hence $x_1 = \frac{2}{3}x_2 - \frac{4}{3}x_3$. Moreover $x_2$ and $x_3$ are free, and therefore the general solution to the system is:

$$\begin{cases} x_1 = \frac{2}{3}x_2 - \frac{4}{3}x_3 \\ x_2 \quad \text{free} \\ x_3 \quad \text{free} \end{cases}$$

**Note:** Try to simplify things whenever you can. For example, in the matrix, divide the second row by 3 and the third row by 2, and *then* start your row-reduction process!

**1.2.15, 1.2.23, 1.2.26.** In each of the problems, the following fact will help you solve the problem:

**Fact:** A system is consistent if and only if in the row echelon form of the augmented matrix there is no row of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & b \end{bmatrix}$$

Where $b \neq 0$.

For **1.2.23**, **1.2.26**, it'll help to draw a picture of what the matrix in question looks like.

For **1.2.26**, try out a concrete example to convince you of this! Can you solve for $z$? If yes, can you solve for $y$? Finally, can you solve for $x$?

**1.2.30.** Underdetermined means 'fewer equations than unknowns'. Find two equations in three unknowns which give you a contradiction, such as $0 = 1$. The easiest way to do this is to write one equation, and then rewrite the same equation, but with a different number on the right.

## Section 1.3: Vector equations

**1.3.7.** Here's a cool trick! **Any** vector in $\mathbb{R}^2$ is a linear combination of two linearly independent vectors! So the answer is immediately $\boxed{\text{yes}}$

**1.3.11, 1.3.15.** Determine if/when the equation $A\mathbf{x} = \mathbf{b}$ has a solution or not (where $A$ is the matrix whose columns are the $\mathbf{a_i}$)

**1.3.22.** All you have to find is an inconsistent system! For example:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

**1.3.25. Careful!** A set is not the same as the span of a set. In particular, $\mathbf{b}$ is not in $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ because it is not equal to either of those vectors. However, it might be in the span of those 3 vectors! Also, for $(c)$, remember that $\mathbf{a_1}$ is always in the span of $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$.

## Section 1.4: The matrix equation $Ax = b$

**1.4.11.** The augmented matrix becomes:

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 1 & 5 & 2 & 4 \\ -3 & -7 & 6 & 12 \end{bmatrix}$$

Now row-reduce: Subtract the first row from the second, and add 3 times the first row to the third:

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 2 & 6 & 6 \\ 0 & 2 & -6 & 6 \end{bmatrix}$$

Divide the second row and the third row by 2:

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & -3 & 3 \end{bmatrix}$$

Subtract the second row from the third:

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -6 & 0 \end{bmatrix}$$

Divide the third row by $-6$:

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Add 4 times the third row to the first, and subtract 3 times the third row from the second:

$$\begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Subtract 3 times the second row from the first:

$$\begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This gives us:

$$\begin{cases} x_1 = -11 \\ x_2 = 3 \\ x_3 = 0 \end{cases}$$

That is:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 0 \end{bmatrix}$$

**1.4.17, 1.4.18.** Row-reduce! Also, use Theorem $4(d)$ on page 45.

**1.4.29.** The easiest way to do this is find a matrix in row-echelon form that has this property, and then just interchange two rows! For example, the following matrix works:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

## Section 1.5: Solution sets of linear systems

**1.5.14.** The line that goes through $\begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}$ and with 'slope' $\begin{bmatrix} 5 \\ -2 \\ 5 \\ 1 \end{bmatrix}$

**1.5.24.**

(a) **F** ($\mathbf{x} = \mathbf{0}$ is always a solution)
(b) **F** (I leave it up to you to come up with such an equation)
(c) **T**
(d) **T** (Because then $\mathbf{b} = A\mathbf{0} = \mathbf{0}$)

**1.5.29.** Nontrivial means $\mathbf{x} \neq \mathbf{0}$. The best way to do this is to draw a picture of what the reduced-echelon form of the matrix looks like! Also, for $(b)$, if one of the rows of $A$ is a row of zeros, then the equation $A\mathbf{x} = \mathbf{b}$ has no solution (for some $\mathbf{b}$).

## Section 1.7: Linear independence

**1.7.1, 1.7.5.** Row-reduce (after putting everything in a matrix, if necessary). If you get $n$ pivots, then the set is linearly independent. Else, it's linearly dependent.

**1.7.5.** We want to solve the system $A\mathbf{x} = \mathbf{0}$. The augmented matrix becomes:

$$\begin{bmatrix} 0 & -3 & 9 & 0 \\ 2 & 1 & -7 & 0 \\ -1 & 4 & -5 & 0 \\ 1 & -4 & -2 & 0 \end{bmatrix}$$

Row-reducing until the matrix is in REF (DO IT!), we get:

$$\begin{bmatrix} 1 & -4 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Fast way: There are 3 pivots in the coefficient matrix, and hence as many pivots as columns (in the coefficient matrix), and hence the vectors are $\boxed{\text{linearly independent}}$.

Slow way: Row-reducing further until we get the RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

But this implies that $x = 0, y = 0, z = 0$, and hence $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Therefore, the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and therefore the vectors are $\boxed{\text{linearly independent}}$

**Note:** It's **VERY** important that you row-reduce until you get the REF. Otherwise you **CANNOT** conclude how many pivots (or free variables) there are. There's no way around this, any other way is considered incorrect!

**1.7.11.** Row-reduce!

**1.7.17.** A set with the zero-vector is always linearly dependent.

**1.7.21.**

(a) **F** (the equation $A\mathbf{x} = \mathbf{0}$ **always** has the trivial solution, no matter what the columns of $A$ look like!)
(b) **F** (for example, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$ doesn't satisfy this! The correct statement should be: there is **some** vector such that $\cdots$)
(c) **T** (in other words, 5 vectors in $\mathbb{R}^4$ are linearly dependent)
(d) **T** (otherwise the set would be linearly independent)

**1.7.23.** This matrix can only have one or no pivots (in the last case, the matrix is the zero-matrix). This is because if the matrix has 2 pivots, the columns would be linearly independent.

**1.7.33.** Remember that a set is linearly dependent if there's a relationship between the vectors in the set. Also, a set with the zero vector is always linearly dependent.

**1.7.36. FALSE** (give me explicit examples of vectors such that $\mathbf{v_1} = \mathbf{v_2}$ and $\mathbf{v_3}$ linearly independent from $\mathbf{v_1}$ and $\mathbf{v_2}$! The point is for linear independence, you have to consider the set as a whole!)

## Section 1.8: Introduction to Linear Transformations

**1.8.3, 1.8.9.** Just solve the equation $A\mathbf{x} = \mathbf{b}$, where in 1.8.9, $\mathbf{b}$ is the zero vector!

**1.8.15.** $T$ is just reflection across the line $y = x$.

**1.8.19.** Use the fact that:

$$\begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\mathbf{e_1} - 3\mathbf{e_2}$$

**1.8.21.**

(a) **T** (it's a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ with special properties)
(b) **F** (the domain is $\mathbb{R}^5$)
(c) **T**
(d) **T** (for **NOW**; in Math 110 you'll see some linear transformations which don't have matrices)
(e) **T** (to get additivity, take $c_1 = c_2 = 1$, to get scalar multiplication, take $c_1 = c, c_2 = 0$)

**1.8.33.** What is $T(0,0,0)$?

**1.8.36.** $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$

## Section 1.9: The matrix of a linear transformation

For **all** of those questions, all you need to find is $T(\mathbf{e_1}), T(\mathbf{e_2}), \cdots$ and group the terms in a matrix!

**1.9.9.** All you need to determine is what happens to $(1,0)$ and $(0,1)$:

If you reflect the point $(1,0)$ through the $x_1-$axis, then you get $(1,0)$ (nothing happens), and if you rotate $(1,0)$ by $-\frac{\pi}{2}$ radians, you get $(0,-1)$, and hence:

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

If you reflect the point $(0,1)$ through the $x_1-$axis, then you get $(0,-1)$, and if you rotate $(0,-1)$ by $-\frac{\pi}{2}$ radians, you get $(-1,0)$, and hence:

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

And putting the two columns together, you get that the matrix of $T$ is $A$, where:

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

**1.9.15.** The first column is given by $T(1,0,0)$, etc.

**1.9.23.**

(a) **T** (in other words, if you know $T(\mathbf{e_1}), T(\mathbf{e_2}), \cdots, T(\mathbf{e_n})$, you know $T$)
(b) **T** (see example 3)
(c) **F** (the composition of two linear transformations is a linear transformation, see chapter 4)
(d) **F** (onto means every vector in $\mathbb{R}^m$ is in the image of $T$)
(e) **F** (let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, then the columns of $A$ are linearly independent, and hence $T$ is one-to-one by theorem $12b$)

**1.9.24.**

(a) **F**
(b) **T**
(c) **T**
(d) **T**
(e) **F**

## Section 2.1: Matrix operations

Remember the rule $(m \times n) \bullet (n \times p) = (m \times p)$.

**2.1.11.** $D = 2I$ works!

**2.1.15.**

(a) **F** (oh, life would be awesome if this was true! But $\mathbf{a_1 a_2}$ doesn't even make sense!)
(b) **F** (the columns of **A** using weights from the column of **B**)
(c) **T**
(d) **T**
(e) **F** (in the *reverse* order, $(AB)^T = B^T A^T$)

**2.1.23.** Multiply the equation $A\mathbf{x} = \mathbf{0}$ by $C$.

**2.1.27.** First figure out the size of your matrix.

## Section 2.2: The inverse of a matrix

**2.2.1.** Use theorem 4.

**2.2.9.** All statements are true, except for $(b)$, because $(AB)^{-1} = B^{-1}A^{-1}$ and $(c)$ (should be $ad - bc \neq 0$, take $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ as an example)

**2.2.13.** Suppose $AB = AC$, where $A$ is invertible. Then, multiplying both sides to the left by $A^{-1}$, we get:

$$\begin{aligned} A^{-1}(AB) &= A^{-1}(AC) \\ (A^{-1}A)B &= (A^{-1}A)C && \text{(by associativity)} \\ IB &= IC && \text{(because } A^{-1}A = I\text{, by definition of inverse)} \\ B &= C && \text{(because } ID = I \text{ for every matrix } D\text{, by definition of } I\text{)} \end{aligned}$$

And therefore $B = C$.

The result is not true in general, because if you take $A = O$ (the zero-matrix), then $AB = OB = O$, and $AC = OC = O$, and hence $AB = AC$, but $B \neq C$.

**2.2.21.** In other words, you have to show that the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$ (think about this in terms of linear combinations of the columns of A). For that, multiply the equation $A\mathbf{x} = \mathbf{0}$ by $A^{-1}$.

**2.2.38.**

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The best way to get $D$ is by using the equation $AD = I_2$ and guessing!

$C$ cannot exist, because otherwise $A$ would be invertible, and in particular its columns would be linearly independent, which is bogus!

## Section 2.3: Characterizations of invertible matrices

For the first few problems, row-reduction is the key!

**2.3.11.** Just look at theorem 8. If one of those statements holds, then all of them hold!

**2.3.13.** Only if all the entries on the diagonal are nonzero! See theorem $8(c)$

**2.3.14.** No! See theorem $8(h)$

**2.3.19.** It has at least one solution for every $\mathbf{b}$ (in fact, exactly one solution), see Theorem $8(g)$.

**2.3.30.** Since $T\mathbf{x} = A\mathbf{x}$ is not one-to-one, by condition $(f)$ of the Invertible Matrix Theorem, $A$ is not invertible. Hence by condition $(i)$ of the Invertible matrix theorem, $T\mathbf{x} = A\mathbf{x}$ cannot map $\mathbb{R}^n$ onto $\mathbb{R}^n$.

**Note:** This is one of the special features of $\mathbb{R}^n$ and about linear transformations! You cannot expect this result to be true in general!

## Section 2.6: Subspaces of $\mathbb{R}^n$

**2.6.3.** Not closed under addition

**2.6.5.** One way to do this is to group the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ together in a matrix $A$, and solve $A\mathbf{x} = \mathbf{w}$.

**2.6.7.**

(a) 3
(b) Infinitely many of them! (but in a sense, you'll see that $Col(A)$ is a 2 or 3 dimensional space).
(c) Solve $A\mathbf{x} = \mathbf{p}$

**2.6.9.** Just check whether $A\mathbf{p} = \mathbf{0}$ or not.

**2.6.21.** All statements are **True**, **EXCEPT** $(c)$ (should be $\mathbb{R}^n$)! Notice that in particular $(a)$ is true! The book is just being picky about this, even though it omitted the word 'for each', the statement still remains true (the words 'for each' here are implied)

**2.6.23.** To find $Nul(A)$, solve $A\mathbf{x} = \mathbf{0}$ using the row-echelon form. To find $Col(A)$, notice that the first two columns of $A$ are pivot columns. In particular, a basis for $Col(A)$ is the set of the first two columns of the *original* matrix $A$.

## Section 2.7: Dimension and Rank

**2.7.3, 2.7.5.** Find the coefficients of $\mathbf{x}$ as a linear combination of $\mathbf{b_1}$ and $\mathbf{b_2}$.

**2.7.9.** To find $Nul(A)$, solve $A\mathbf{x} = \mathbf{0}$ using the row-echelon form. To find $Col(A)$, locate the pivot columns of $A$. In particular, a basis for $Col(A)$ is the set of the pivot columns of the *original* matrix $A$.

**2.7.16, 2.7.21.** Use the fact that $\dim(Nul(A)) + \text{rank}(A) = n$

**2.7.23.** For example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**2.7.24.** For example:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

## Section 3.1: Introduction to determinants

Always try to look for a row/column full of zeros! May Bomberman be with you :)

**3.1.9.** Expanding along the third row, and then along the first row, we get:

$$\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix} = (2)(5) \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 10(7-6) = 10$$

**3.1.19, 3.1.21.** What you're asked to do is: compute the determinants of the first matrix and of the second matrix and compare them. Also, explain how to obtain the second matrix from the first using a row-operation!

**3.1.41.** The area of the parallelogram always equals to the determinant of $[\mathbf{u} \ \ \mathbf{v}]$. Use the formula: Area of parallelogram = base × height. The two areas should be the same (by the way, this fact is a very simplified version of Cavalieri's principle)

## Section 3.2: Properties of determinants

**3.2.11.** What they mean is: First calculate the determinant by expanding along the second column, and then evaluate the resulting sub-determinants using row-reduction!

**3.2.19. DO NOT EVALUATE THE DETERMINANT!** Use row-reduction! In particular, notice that to obtain the determinant in the problem, all you have to do is multiply the second row of the original matrix by 2, and then add the first row to the second row! Hence, the answer should be $2 \times 7 = 14$.

**3.2.21.** A matrix $A$ is invertible if and only if $det(A) \neq 0$.

**Note:** From now on, I'm only giving the answer to the T/F questions! I leave it up to you to explain why the result is true or false.

**3.2.27.**

(a) **T** ? I think the book uses the term 'row-replacement' to mean: " add $k$ times a row to another row".
(b) **F** (not true for *any* echelon form, what about the reduced row-echelon form?)
(c) **T**
(d) **F**

**3.2.31, 3.2.33, 3.2.34, 3.2.35.** All you need to use is the fact that $det(AB) = det(A)det(B)$.

**3.2.34.**

$$\det\left(PAP^{-1}\right) = \det(P)\det(A)\det\left(P^{-1}\right) = \cancel{\det(P)}\det(A)\frac{1}{\cancel{\det(P)}} = \det(A)$$

## Section 3.3: Cramer's rule, volume, and linear transformations

For all those problems, all you need to do is imitate the techniques presented in the book.

**3.3.7.** The system has a unique solution iff $det(A) \neq 0$ (because that's equivalent to saying that $A$ is invertible)

**3.3.21.** The only thing that makes this difficult is that the parallelogram is not centered at $(0,0)$. To make it centered at $(0,0)$, just shift it to the right by one unit! The area stays the same anyway!

**3.3.32.**

(a) Define $T$ by $T(e_i) = v_i$. Notice that this is enough to define $T$, because $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$.

(b) Now just use $Vol(S') = det(T)Vol(S)$. The volume of $S$ is $\frac{1}{3} \times \frac{1}{2}(1 \times 1) \times 1 = \frac{1}{6}$, because all of its three lengths are equal to 1. As for $det(T)$, that's just the determinant of the matrix whose columns are $v_1, v_2, v_3$.

## Section 4.1: Vector spaces and subspaces

Remember the three techniques of showing whether something is a vector space or not!

(1) Trick 1: Show it is not a vector space by finding an explicit property which does not hold
(2) Trick 2: Show it is a subspace of a (known) vector space
(3) Trick 3: Express it in the form *Span* of some vectors.

**4.1.13(c).** For $(c)$, to show $\mathbf{w}$ is in the subspace or not, all you have to show is whether the system $A\mathbf{x} = \mathbf{w}$ is consistent or not (where $A$ is the matrix whose columns are the $\mathbf{v}_i$).

**4.1.24.**

(a) **T** (this is important to remember!!! A vector isn't a list of numbers any more, it could be anything, even a function!)
(b) **T**
(c) **T** (of itself!)
(d) **F**
(e) **T** (again, the textbook might give you a different answer, but I agree that this is weirdly phrased! What they mean is: If $\mathbf{u}, \mathbf{v}$ is in $H$, then $\mathbf{u} + \mathbf{v}$ is in $H$).

**4.1.32.** This is a bit tricky! Remember that $H \cap K$ is the set of vectors that is both in $H$ and in $K$. Here's the proof that $H \cap K$ is closed under addition (hopefully that'll inspire you to do the rest):

Suppose $\mathbf{u}$ and $\mathbf{v}$ are in $H \cap K$. Then $\mathbf{u}$ and $\mathbf{v}$ are in $H$, so is $\mathbf{u} + \mathbf{v}$ (since $H$ is a subspace). Also, since $\mathbf{u}$ and $\mathbf{v}$ are in $K$, so is $\mathbf{u} + \mathbf{v}$ (since $K$ is a subspace). Hence $\mathbf{u} + \mathbf{v}$ is both in $H$ and $K$, hence $\mathbf{u} + \mathbf{v}$ is in $H \cap K$.

As for the fact that the union of two subspaces is not a subspace, take $H$ to be the $x-$axis, and $K$ to be the $y-$ axis. Then $(1,0)$ and $(0,1)$ are both in the union, but $(1,1)$ is not.

## Section 4.2: Nullspaces, Column Spaces, and Linear Transformations

This is very similar to what you've been doing in sections 2.8 and 2.9. See also the tricks I gave in the beginning of section 4.1.

**4.2.7.** It's not a subspace of $\mathbb{R}^3$ because $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in it, since $0+0+0 = 0 \neq 2$.

**4.2.23.** Is $\mathbf{w}$ a linear combo of the columns of $A$? Is $A\mathbf{w} = \mathbf{0}$?

**4.2.25.**

(a) **T**
(b) **F**
(c) **T**
(d) **T** (the book might say **F**, if it is pedantic about the fact that it didn't say 'for all $\mathbf{b}$')
(e) **T**
(f) **T**

## Section 4.3: Linearly independent sets, bases

Remember that a basis is a linearly independent set which spans the whole space! Equivalently, as set is a basis if the corresponding matrix $A$ is invertible.

**4.3.7.** It cannot span $\mathbb{R}^3$ because we have 2 vectors in $\mathbb{R}^3$ (and hence it cannot be a basis). To check linear independence, just ask yourself: is the second vector a multiple of the first?

**4.3.13.** To find $Col(A)$, see where the pivot columns are, and then go back to $A$ and choose precisely those columns (here the first and second column of $A$)

**4.3.21.**

(a) **FALSE** (consider $V = \mathbb{R}^2$, then $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent).

(b) **FALSE** ($\{\mathbf{b_1}, \cdots, \mathbf{b_p}\}$ *could* be linearly dependent!)

(c) **TRUE** (by the Invertible Matrix Theorem)

(d) **FALSE** (it's the *smallest* spanning set, see page 200)

(e) **FALSE** (row-operations *preserve* linear independence relationships among the columns, see page 199)

**4.3.32.** I love this problem!!! We are given that there exist $c_1, \cdot, c_p$, not all 0, such that:

$$c_1 T(v_1) + \cdots + c_p T(v_p) = \mathbf{0}$$

By linearity of $T$, this becomes:

$$T(c_1 v_1 + \cdots + c_p v_p) = \mathbf{0} = T(\mathbf{0})$$

But because $T$ is one-to-one, this implies:

$$c_1 v_1 + \cdots + c_p v_p = \mathbf{0}$$

Since the $c_1, \cdot, c_p$ are not all 0, this shows that the vectors $v_1, \cdot, v_p$ are linearly dependent.

Make sure to understand every step of this (and enjoy how near this is!)

## Section 4.4: Coordinate Systems

Remember: It's easier to figure out $\mathbf{x}$ once we know $[\mathbf{x}]_{\mathcal{B}}$ than the reverse. Also, remember that the change of coordinates matrix is just the matrix whose columns are the elements in $\mathcal{B}$.

It takes a code as its input and tells you which vector corresponds to that code. Its inverse matrix does what you usually want: It produces the coordinates of $\mathbf{x}$

**4.4.15.**

(a) **T**
(b) **F** (it's $\mathbf{x} = P_{\mathcal{B}} [\mathbf{x}]_{\mathcal{B}}$ )
(c) **F** (it's $P_2$ and $\mathbb{R}^3$ which are isomorphic)

**4.4.19.** This is also kind of cute. Let $S = \{v_1, \cdots, v_n\}$

Span:

Let $\mathbf{x}$ be an arbitrary vector in $V$. Then we know that $\mathbf{x}$ has a representation as a linear combination of elements of $S$, that is, there exist $c_1, \cdots, c_n$ such that:

$$\mathbf{x} = c_1 v_1 + \cdots + c_n v_n$$

And therefore $\mathbf{x}$ is in the span of $S$, and therefore $S$ spans $V$ (since $\mathbf{x}$ was arbitrary in $V$).

Linear independence:

Suppose there exist constants $c_1, \cdots, c_n$ such that:

$$c_1 v_1 + \cdots + c_n v_n = \mathbf{0}$$

However, notice that we can write $\mathbf{0} = 0v_1 + \cdots + 0v_n$, and therefore we found *two* ways of writing $\mathbf{0}$ as a linear combinations of elements in $S$. But since there is only one way of writing $\mathbf{0}$ as a linear combination of elements in $S$ (by assumption), it follows that the $c_i$ *must* all equal to 0. Therefore $c_1 = \cdots = c_n = 0$, and therefore $S$ is also linearly independent.

**4.4.27.** Use the basis $\mathcal{B} = \left\{1, t, t^2, t^3\right\}$, and compute the coordinates of the 4 polynomials. Then the polynomials are linearly independent if and only if their corresponding vectors are linearly independent!

## Section 4.5: The dimension of a vector space

**4.5.3, 4.5.7, 4.5.11.** First express the subspace as the span of some vectors, and then use the following useful trick:

**Useful trick:** To find a basis of a collection of vectors, form the matrix $A$ whose columns are the vectors, and all you need to do is to find a basis for $Col(A)$. In particular, the dimension of the subspace is the dimension of $Col(A)$ (which is the number of pivots).

**4.5.26.** Suppose $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}\}$ is a basis for $H$. What two things can you say about $\mathcal{B}$? Then use the Basis theorem (Theorem 12).

**4.5.27.** Find an infinite linearly independent set in $\mathbb{P}$. For example, $\left\{1, x, x^2 \cdots\right\}$ works!

## Section 4.6: The rank of a matrix

Remember that the rank of $A$ is just $dim(Col(A))$. It is also equal to $dim(Row(A))$ and to $Rank(A^T)$ and to the number of pivots of $A$.

**4.6.1, 4.6.5, 4.6.9, 4.6.15.** Use the equation $dim(Nul(A)) + Rank(A) = n$. Also, $rank(A)$ is largest when $Nul(A)$ is smallest.

**4.6.9.** By the Rank-Theorem, we have:

$$\dim(Col(A)) = 6 - \dim(Nul(A)) = 6 - 3 = 3$$

But this does **NOT** imply that $Col(A) = \mathbb{R}^3$, because $Col(A)$ is in $\mathbb{R}^4$, so it **CANNOT** equal to $\mathbb{R}^3$.

**4.6.22.** This question is just meant to confuse you with words! All that it says is that if you have an $10 \times 12$ matrix, could $Nul(A)$ eveyr be $1-$dimensional? Use rank-nullity to argue that it cannot.

**4.6.33.** I urge you to do 4.6.32 before, it makes this much easier! The point is that if $A$ has rank 1, then all its columns are multiples of the first column. In particular, let $\mathbf{v}$ be the list of the coefficients. For example, if $A = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}$, then let $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, because the second column is $-3$ times the first one and the third column is 4 times the first one.

If the first column of $A$ is zero, try the second column. If the second column is zero, try the third column. If neither of those hold, then $A$ is the zero matrix, which does not have rank 1.

## Section 4.7: Change of basis

Remember: To change coordinates $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$ from $\mathcal{B}$ to $\mathcal{C}$, just express the vectors in $\mathcal{B}$ in terms of the vectors in $\mathcal{C}$

**4.7.3.** $(ii)$, because $P$ is just $\mathcal{W} \overset{P}{\leftarrow} \mathcal{U}$, so $P$ goes from $\mathcal{U}$ to $\mathcal{W}$.

**4.7.5.** $(i)$, because $P$ is just $\mathcal{A} \overset{P}{\leftarrow} \mathcal{D}$, so $P$ goes from $\mathcal{D}$ to $\mathcal{A}$.

**4.7.9.** First, we want to find $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$. For this, row-reduce:

$$\left[ \mathcal{C} \mid \mathcal{B} \right] = \begin{bmatrix} 2 & -2 & 4 & 8 \\ 2 & 2 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Therefore:

$$\mathcal{C} \overset{P}{\leftarrow} \mathcal{B} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$$

And then:

$$\mathcal{B} \overset{P}{\leftarrow} \mathcal{C} = \left( \mathcal{C} \overset{P}{\leftarrow} \mathcal{B} \right)^{-1} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & -1 \end{bmatrix}$$

**4.7.11.**

(a) **F**
(b) **T**

**4.7.13.** Notice that we have $[\mathbf{x}]_{\mathcal{C}} = P\,[\mathbf{x}]_{\mathcal{B}}$, where $P$ is the matrix you found, and therefore $[\mathbf{x}]_{\mathcal{B}} = P^{-1}\,[\mathbf{x}]_{\mathcal{C}}$.

## Section 5.1: Eigenvalues and eigenvectors

Remember: To find the eigenvalues, calculate $det(A - \lambda I)$ and find the zeros of the resulting polynomial. To find a basis for the eigenspaces, find $Nul(A - \lambda I)$ for each eigenvalue $\lambda$ that you found! Also, you should never get $Nul(A - \lambda I) = \{\mathbf{0}\}$

**5.1.1, 5.1.5.** Calculate $A\mathbf{v}$, where $A$ is the given matrix and $\mathbf{v}$ is the given vector.

**5.1.13, 5.1.17.** Remember that the determinant of an upper-triangular matrix is just the product on the entries of the diagonal! (so you can literally 'read' off the eigenvalues)

**5.1.21.**

(a) **F** ($\mathbf{x}$ has to be nonzero)
(b) **T**
(c) **T**
(d) **T** (depending on what you mean by easy and hard :) )
(e) **F**

## Section 5.2: The characteristic equation

**5.2.11.** Using Bomberman and expanding along the first row, we get that the characteristic polynomial of $A$ is:

$$\begin{aligned}\det(\lambda I - A) = &\begin{vmatrix} \lambda-3 & 0 & 0 \\ -2 & \lambda-1 & -4 \\ -1 & 0 & \lambda-4 \end{vmatrix} \\ =&(\lambda-3)\begin{vmatrix} \lambda-1 & -4 \\ 0 & \lambda-4 \end{vmatrix} \\ =&(\lambda-3)(\lambda-1)(\lambda-4) \\ =&(\lambda-1)(\lambda-3)(\lambda-4)\end{aligned}$$

**5.2.15.** Remember that the determinant of an upper/lower-triangular matrix is just the product on the entries of the diagonal!

**5.2.19.** Just plug in $\lambda = 0$.

**5.2.21.**

(a) **F**
(b) **F**
(c) **T**
(d) **F** ($-5$ is an eigenvalue)

## Section 5.3: Diagonalization

**5.3.1, 5.3.3.** If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$

**5.3.5.** The eigenvalues are just the diagonal entries of $D$, and the eigenvectors are the corresponding columns of $P$

**5.3.7, 5.3.11, 5.3.17.** All you have to do is to find $D$ and $P$ so that $A = PDP^{-1}$. To find $D$, find the eigenvalues. To find $P$, find the eigenvectors, and put them together in a matrix.

**5.3.21.**

(a) **FALSE** ($D$ has to be **diagonal**!)

(b) **TRUE** (Theorem 5; Let $D$ be the matrix of eigenvalues, and $P$ be the matrix of corresponding eigenvectors)

(c) **FALSE** (Notice that we didn't say *distinct* eigenvalues. It *is* true that if $A$ is diagonalizable, then by Theorem 7$b$, $A$ has $n$ eigenvalues *including multiplicities*; but for example the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has eigenvalue 1 with multiplicity 2 (so 2 eigenvalues counting multiplicity) but $A$ is not diagonalizable)

(d) **FALSE** (Take $A$ to be the $O$ matrix. Then $A$ is not invertible, but $A$ is diagonalizable because it's diagonal)

## Section 5.4: Eigenvectors and linear transformations

**5.4.3.** Remember that $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = (1, 0, 0)$ etc. To find the matrix in (c), just put the answers you find in (b) together in a matrix. It's that easy!

**5.4.7.** For every polynomial $p = 1, t, t^2$, calculate $T(p)$, and express your answer in terms of $1, t, t^2$. The coefficients give you each column of your matrix.

**5.4.15.** Find the eigenvectors of $A$ (that's sort of the point of this section)

## Section 5.5: Complex eigenvalues

**5.5.1, 5.5.3.** Just use the same technique you usually use to find eigenvalues and eigenvectors!

**5.5.7.** First, calculate $r = \sqrt{det(A)}$ (or take the length of the first row of $A$). Then factor out $r$ from $A$ and recognize the resulting matrix as a rotation matrix, i.e. find $\phi$ such that the remaining matrix equals to $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$

**5.5.13, 5.5.15.** First, find the eigenvalues of $A$, and pick one of them. Then the first **ROW** of $C$ consists the real and imaginary parts of the eigenvalue you picked. Then remember that the diagonal entries of $C$ are the same, and the other entries are opposites of each other. Finally, to get $P$, find an eigenvector corresponding to the eigenvalue you picked, and then the columns of $P$ are the real and imaginary parts of that eigenvector!

## Section 6.1: Inner products, lengths, and orthogonality

**6.1.7.** Divide the vector by its length!

**6.1.19.**

(a) **T**
(b) **T**
(c) **T** (see pages 279-280)
(d) **F** (Vectors in $Col(A)$ are orthogonal to vectors in $Nul(A^T)$, by Theorem 3)
(e) **T** (this is **1.** at the bottom of page 280)

**6.1.22.** $u \cdot u = u_1^2 + u_2^2 + u_3^2 \geq 0$ (as a sum of squares), and this is $= 0$ if and only if all the $u_i = 0$ (and hence $u = \mathbf{0}$), because a sum of squares is 0 if and only if each component is 0.

**6.1.24.**

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2$$

$$= (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v})$$

(by the definition of $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$)

$$= \mathbf{u} \cdot (\mathbf{u}+\mathbf{v}) + \mathbf{v} \cdot (\mathbf{u}+\mathbf{v}) + \mathbf{u} \cdot (\mathbf{u}-\mathbf{v}) + (-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v})$$

(because $\mathbf{x} \cdot (\mathbf{y}+\mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z}$)

$$= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot (-\mathbf{v}) + (-\mathbf{v}) \cdot \mathbf{u} + (-\mathbf{v}) \cdot (-\mathbf{v})$$

(because $\mathbf{x} \cdot (\mathbf{y}+\mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$ and $\mathbf{y} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{y}$, for all $\mathbf{x}, \mathbf{y}, \mathbf{z}$)

$$= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}$$

(because $(c\mathbf{x})\cdot\mathbf{y} = \mathbf{x}\cdot(c\mathbf{y}) = c\,(\mathbf{x}\cdot\mathbf{y})$ for all $\mathbf{x},\mathbf{y}$)

$$= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}$$

(because $\mathbf{y}\cdot\mathbf{x} = \mathbf{x}\cdot\mathbf{y}$, for all $\mathbf{x},\mathbf{y}$)

$$= \|\mathbf{u}\|^2 + \cancel{2\mathbf{u}\cdot\mathbf{v}} + \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - \cancel{2\mathbf{u}\cdot\mathbf{v}} + \|\mathbf{v}\|^2$$

(because $\mathbf{x}\cdot\mathbf{x} = \|\mathbf{x}\|^2$, and $\mathbf{x}+\mathbf{x} = 2\mathbf{x}$ for all $\mathbf{x}$)

$$= 2\,\|\mathbf{u}\|^2 + 2\,\|\mathbf{v}\|^2$$

## Section 6.2: Orthogonal sets

Remember: A set $\mathcal{B}$ is orthogonal if for every pair of distinct vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}\cdot\mathbf{v} = 0$. It is orthonormal if it is orthogonal and every vector has length 1. An orthogonal set can be made orthonormal by dividing every vector by its length.

**6.2.7.**

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{\mathbf{x}\cdot\mathbf{u_1}}{\mathbf{u_1}\cdot\mathbf{u_1}} \\ \frac{\mathbf{x}\cdot\mathbf{u_2}}{\mathbf{u_2}\cdot\mathbf{u_2}} \end{bmatrix}$$

**6.2.9.**

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{\mathbf{x}\cdot\mathbf{u_1}}{\mathbf{u_1}\cdot\mathbf{u_1}} \\ \frac{\mathbf{x}\cdot\mathbf{u_2}}{\mathbf{u_2}\cdot\mathbf{u_2}} \\ \frac{\mathbf{x}\cdot\mathbf{u_3}}{\mathbf{u_3}\cdot\mathbf{u_3}} \end{bmatrix}$$

**6.2.13, 6.2.15.** The formula for orthogonal projection of $\mathbf{y}$ on the line spanned by $\mathbf{u}$ is:

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

Then you can write $\mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}})$ Notice that $\hat{\mathbf{y}}$ is in the span of $\mathbf{u}$, whereas $(\mathbf{y} - \hat{\mathbf{y}})$ is orthogonal to $\mathbf{u}$.

The distance between $\mathbf{u}$ and $L$ is then $\|\mathbf{y} - \hat{\mathbf{y}}\|$

**6.2.23.**

(a) **T** (Take $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^3$ )
(b) **T** (You just use the formula in Theorem 5 on page 285; THIS is what makes orthogonal sets so awesome!)
(c) **F** (They're still orthogonal, this is because $\frac{u}{\|u\|} \cdot \frac{v}{\|v\|} = \frac{1}{\|u\|\|v\|} u \cdot v = 0$ if $u$ and $v$ are orthogonal)
(d) **F** (in this course, we assume that orthogonal matrices must be **square**[1])
(e) **F** (it's $\|\mathbf{y} - \hat{\mathbf{y}}\|$ which gives that distance)

## Section 6.3: Orthogonal projection

Here are all the basic facts that you'll need:

(1) If $W = Span\{\mathbf{u_1}, \mathbf{u_2} \cdots \mathbf{u_k}\}$, then the orthogonal projection of $\mathbf{y}$ onto $W$ is:

$$\hat{\mathbf{y}} = \left( \frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \right) \mathbf{u_1} + \left( \frac{\mathbf{y} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}} \right) \mathbf{u_2} + \cdots + \left( \frac{\mathbf{y} \cdot \mathbf{u_k}}{\mathbf{u_k} \cdot \mathbf{u_k}} \right) \mathbf{u_k}$$

(2) Then $\hat{\mathbf{y}}$ is in $W$, $y - \hat{\mathbf{y}}$ is in $W^{\perp}$ (that is, orthogonal to $W$).
(3) $\mathbf{y} = (\hat{\mathbf{y}}) + (y - \hat{\mathbf{y}})$, which decomposes $\mathbf{y}$ as a sum of two vectors, one in $W$ and the other one orthogonal to $W$.
(4) $\hat{\mathbf{y}}$ is the closest point to $\mathbf{y}$ in $W$.
(5) $\|y - \hat{\mathbf{y}}\|$ is the smallest distance between $\mathbf{y}$ and $W$.

[1] but not in other courses, beware!

**6.3.11.**

$$\begin{aligned}
\hat{\mathbf{y}} &= \left(\frac{\mathbf{y}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} + \left(\frac{\mathbf{y}\cdot\mathbf{v_2}}{\mathbf{v_2}\cdot\mathbf{v_2}}\right)\mathbf{v_2} \\
&= \left(\frac{9+1-5+1}{9+1+1+1}\right)\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} + \left(\frac{3-1+5-1}{1+1+1+1}\right)\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} \\
&= \left(\frac{6}{12}\right)\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} + \left(\frac{6}{4}\right)\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} \\
&= \left(\frac{1}{2}\right)\begin{bmatrix}3\\1\\-1\\1\end{bmatrix} + \left(\frac{3}{2}\right)\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} \\
&= \begin{bmatrix}\frac{3}{2}+\frac{3}{2}\\ \frac{1}{2}-\frac{3}{2}\\ -\frac{1}{2}+\frac{3}{2}\\ \frac{1}{2}-\frac{3}{2}\end{bmatrix} \\
&= \begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}
\end{aligned}$$

**6.3.21.**

(a) **T**
(b) **T**
(c) **F**
(d) **T**
(e) **T**

## Section 6.4: The Gram-Schmidt process

Use the formula given in Theorem 11. To get an *orthonormal* basis, just divide every vector at the end by its length. At every step, it's helpful to multiply your vector by a scalar to avoid fractions. This is ok, because you'll normalize them at the end anyway!

**6.4.9.** Just apply Gram-Schmidt to the columns of $A$.

**6.4.17.**

(a) **F** (Although the set would be orthogonal, multiplying by $c = 0$ wouldn't give an orthogonal *basis*)
(b) **T** (by (1) in Theorem 11)
(c) **T** (if $A = QR$, then $Q^T A = Q^T QR = R$, since $Q$ has orthonormal columns)

## Section 6.5: Least squares problems

Here's the general procedure to solve least-squares problems: To solve $A\mathbf{x} = \mathbf{b}$ in the least-squares sense, multiply both sides by $A^T$, and solve the (easier) equation $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$. Your solution $\hat{\mathbf{x}}$ is called the least-squares solution. The least squares error is $\|A\hat{\mathbf{x}} - \mathbf{b}\|$.

**6.5.9.** The orthogonal projection of $\mathbf{b}$ is given by $\hat{\mathbf{b}} = \frac{\mathbf{b}\cdot a_1}{a_1\cdot a_1}a_1 + \frac{\mathbf{b}\cdot a_2}{a_2\cdot a_2}a_2$, where $a_1, a_2$ are the columns of $A$. Then all you have to do is solve $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$

**6.5.11.**

(a) Denote the columns of $A$ by $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$

$$\begin{aligned}
\hat{\mathbf{b}} &= \left(\frac{\mathbf{b}\cdot\mathbf{a_1}}{\mathbf{a_1}\cdot\mathbf{a_1}}\right)\mathbf{a_1} + \left(\frac{\mathbf{b}\cdot\mathbf{a_2}}{\mathbf{a_2}\cdot\mathbf{a_2}}\right)\mathbf{a_2} + \left(\frac{\mathbf{b}\cdot\mathbf{a_3}}{\mathbf{a_3}\cdot\mathbf{a_3}}\right)\mathbf{a_3} \\
&= \frac{36}{54}\begin{bmatrix}4\\1\\6\\1\end{bmatrix} + \frac{0}{27}\begin{bmatrix}0\\-5\\1\\-1\end{bmatrix} + \frac{9}{27}\begin{bmatrix}1\\1\\0\\-5\end{bmatrix} \\
&= \frac{2}{3}\begin{bmatrix}4\\1\\6\\1\end{bmatrix} + 0\begin{bmatrix}0\\-5\\1\\-1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\1\\0\\-5\end{bmatrix} \\
&= \begin{bmatrix}3\\1\\4\\-1\end{bmatrix}
\end{aligned}$$

(b) <u>Standard-way:</u>

Solve $A\widetilde{\mathbf{x}} = \hat{\mathbf{b}}$ by row-reduction::

$$\begin{bmatrix}4 & 0 & 1 & 3\\1 & -5 & 1 & 1\\6 & 1 & 0 & 4\\1 & -1 & -5 & -1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 0 & \frac{2}{3}\\0 & 1 & 0 & 0\\0 & 0 & 1 & \frac{1}{3}\\0 & 0 & 0 & 0\end{bmatrix}$$

Therefore:

$$\widetilde{\mathbf{x}} = \begin{bmatrix}\frac{2}{3}\\0\\\frac{1}{3}\end{bmatrix}$$

OMG-way: Notice that in $(a)$, we obtained:

$$\frac{2}{3}\begin{bmatrix}4\\1\\6\\1\end{bmatrix}+0\begin{bmatrix}0\\-5\\1\\-1\end{bmatrix}+\frac{1}{3}\begin{bmatrix}1\\1\\0\\-5\end{bmatrix}=\begin{bmatrix}3\\1\\4\\-1\end{bmatrix}$$

which tells us directly that:

$$A\begin{bmatrix}\frac{2}{3}\\0\\\frac{1}{3}\end{bmatrix}=\hat{\mathbf{b}}$$

therefore the least-squares solution is:

$$\widetilde{\mathbf{x}}=\begin{bmatrix}\frac{2}{3}\\0\\\frac{1}{3}\end{bmatrix}$$

**6.5.17.**

(a) **T**
(b) **T**
(c) **F**
(d) **T**
(e) **T**

## Section 6.7: Inner product spaces

**6.7.1.** Here $\langle \mathbf{x}, \mathbf{y}\rangle = 4u_1v_1 + 5u_2v_2$

**6.7.5, 6.7.7.** Here $\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. And $\|p\| = \sqrt{\langle p, p\rangle}$. Finally, remember that the formula for orthogonal projection remains the same, namely:

$$\hat{q} = \frac{\langle q, p\rangle}{\langle p, p\rangle}p$$

**6.7.11.** Here $\langle p, q\rangle = p(-2)q(-2) + p(-1)q(-1) + p(0)q(0) + p(1)q(1) + p(2)q(2)$.

If we let $p_3 = t^2$, then we have:

$$\hat{p_3} = \frac{\langle p_3, p_0\rangle}{\langle p_0, p_0\rangle}p_0 + \frac{\langle p_3, p_1\rangle}{\langle p_1, p_1\rangle}p_1 + \frac{\langle p_3, p_2\rangle}{\langle p_2, p_2\rangle}p_2$$

**6.7.16.** This is very cute! Notice that:

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v} \quad \text{(since } \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal)}\\ &= \|\mathbf{u}\|^2+\|\mathbf{v}\|^2\\ &= 1+1 \quad \text{(by ortho}\textit{normality}\text{)}\\ &= 2\end{aligned}$$

Therefore $\|\mathbf{u}-\mathbf{v}\|^2 = 2$, and hence $\|\mathbf{u}-\mathbf{v}\| = \sqrt{2}$ (by taking square roots)

## Section 7.1: Diagonalization of Symmetric matrices

**7.1.9.** Remember orthogonal matrices have **orthonormal** columns!

**7.1.17.** First diagonalize the matrix as usual, and then apply Gram-Schmidt to each eigenspace!

**7.1.25.**

(a) T (by theorem 2; this is the most important fact about symmetric matrices!)
(b) T (by theorem 1)
(c) F (take the identity matrix for example)
(d) F ($\mathbf{v}$ has to be a *unit* vector)

## Section 4.2: Homogeneous linear equations: the general solution

**4.2.27.** Linearly dependent, because $\sin(2t) = 2\cos(t)\sin(t)$

**4.2.34.**

(a) Just evaluate the determinant

(b) ($\Rightarrow$) If there is some point $\tau$ where $W = 0$ at $\tau$, then by Lemma 1, $y_1$ and $y_2$ are linearly dependent.
($\Leftarrow$) Suppose that $ay_1(t) + by_2(t) = 0$ for all $t$. Then differentiating this, we get $ay_1'(t) + by_2'(t) = 0$, but then we have:

$$\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But since $W$ is never 0 on $I$, the determinant of the first matrix is nonzero, and hence that matrix is invertible, and hence $a = 0$ and $b = 0$, so $y_1$ and $y_2$ are linearly independent on $I$.

(c) First assume that $y_1 = cy_2$, and calculate $W[y_1, y_2] = W[cy_2, y_2]$ and show it's equal to 0. Then assume that $y_2 = cy_1$ and calculate $W[y_1, y_2] = W[y_1, cy_1]$ and show that you get 0 in both cases.

## Section 4.3: Auxiliary equations with complex roots

The problems should hopefully be pretty straightforward :)

**4.3.21.** The auxiliary equation is $r^2 + 2r + 2 = 0$, which gives:

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

Which means that the general solution to the differential equation is:

$$y(t) = Ae^{-t}\cos(t) + Be^{-t}\sin(t)$$

Plugging in $t = 0$, we get $y(0) = A = 2$, so $\boxed{A = 2}$ and hence:

$$y(t) = 2e^{-t}\cos(t) + Be^{-t}\sin(t)$$

Differentiating, we get:

$$\begin{aligned} y'(t) =& -2e^{-t}\cos(t) + 2e^{-t}(-\sin(t)) - Be^{-t}\sin(t) + Be^{-t}\cos(t) \\ =&(B-2)e^{-t}\cos(t) + (-2-B)e^{-t}\sin(t) \end{aligned}$$

Plugging in $t = 0$, we get:

$y'(0) = (B-2) = 1$, and so $\boxed{B = 3}$.

Therefore our solution is:

$$y(t) = 2e^{-t}\cos(t) + 3e^{-t}\sin(t)$$

**4.3.29(b).** The following fact might be useful:

**Rational roots theorem:** If a polynomial $p$ has a zero of the form $r = \frac{a}{b}$, then $a$ divides the constant term of $p$ and $b$ divides the leading coefficient of $p$.

This helps you 'guess' a zero of $p$. Then use long division to factor out $p$.

## Section 4.4: The method of undetermined coefficients

**4.4.3.** Yes. $\frac{\sin(x)}{e^{4x}} = e^{-4x}\sin(x)$.

**4.4.5.** Yes. $4x\sin^2(x) + 4x\cos^2(x) = 4x$.

**4.4.7.** No. The method of undetermined coefficients only works for **constant-coefficient** linear differential equations, which is not the case because the coefficient of $y''$ is $t$.

**4.4.13.** Guess $y_p(t) = A\cos(3t) + B\sin(3t)$

**4.4.21.** Guess $y_p(t) = (At+B)t^2e^{2t}$

(Always treat the polynomial term separately! You have to guess $t^2e^{2t}$ because the general solution to the homogeneous equation is already $Ae^{2t} + Bte^{2t}$)

**4.4.27.** Guess:

$$y_p(t) = (At^3 + Bt^2 + Ct + D)t\cos(3t) + (Et^3 + Ft^2 + Gt + H)t\sin(3t)$$

(Always treat the polynomial term separately! You have to guess $t\cos(3t)$ and $t\sin(3t)$ because the general solution to the homogeneous equation is already $A\cos(3t) + B\sin(3t)$)

**4.4.31.** Guess:

$$y_p(t) = (At^3 + Bt^2 + Ct + D)te^{-t}\cos(t) + (Et^3 + Ft^2 + Gt + H)te^{-t}\sin(t)$$

(same remark as 27)

## Section 4.5: The superposition principle

**4.5.1(b).** By linearity, the solution is $y(t) = 2y_2(t) - 3y_1(t)$

**4.5.3, 4.5.5.** Use the fact that $y = y_p + y_0$, where $y_0$ is the general solution of the homogeneous equation.

**4.5.9.** Yes, because $(e^t + t)^2 = e^{2t} + 2te^t + t^2$

**4.5.21.** Homogeneous equation:

First of all, the auxiliary equation is $r^2 + 2r + 2 = 0$, which gives:

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

which tells you the general solution of $y'' + 2y' + 2 = 0$ is:

$$y_0(\theta) = Ae^{-\theta}\cos(\theta) + Be^{-\theta}\sin(\theta)$$

Particular solution:

Notice that $e^{-\theta}\cos(\theta)$ is *already* a solution of the homogeneous equation, so we'll have to guess:

$$y_p(\theta) = A\theta e^{-\theta}\cos(\theta) + B\theta e^{-\theta}\sin(\theta)$$

This gives us:

$$\begin{aligned} y_p'(\theta) =& Ae^{-\theta}\cos(\theta) - A\theta e^{-\theta}\cos(\theta) - A\theta e^{-\theta}\sin(\theta) + Be^{-\theta}\sin(\theta) - B\theta e^{-\theta}\sin(\theta) + B\theta e^{-\theta}\cos(\theta) \\ =& (A - A\theta + B\theta)e^{-\theta}\cos(\theta) + (B - B\theta - A\theta)e^{-\theta}\sin(\theta) \end{aligned}$$

And:

$$\begin{aligned} y_p''(\theta) =&(B-A)e^{-\theta}\cos(\theta) - (A - A\theta + B\theta)e^{-\theta}\cos(\theta) - (A - A\theta + B\theta)e^{-\theta}\sin(\theta) \\ &+(-B-A)e^{-\theta}\sin(\theta) - (B - B\theta - A\theta)e^{-\theta}\sin(\theta) + (B - B\theta - A\theta)e^{-\theta}\cos(\theta) \\ =&(B - A - A + \cancel{A\theta} - B\theta + B - B\theta - \cancel{A\theta})e^{-\theta}\cos(\theta) \\ &+(-A + A\theta - \cancel{B\theta} - B - A - B + \cancel{B\theta} + A\theta)e^{-\theta}\sin(\theta) \\ =&(2B - 2A - 2B\theta)e^{-\theta}\cos(\theta) + (-2A - 2B + 2A\theta)e^{-\theta}\sin(\theta) \end{aligned}$$

Plugging those formulas into our equation $y'' + 2y' + 2y = e^{-\theta}\cos(\theta)$, we get:

$$\begin{aligned} y_p'' + 2y_p' + 2y_p &= e^{-\theta}\cos(\theta) \\ \left[(2B - 2A - 2B\theta)e^{-\theta}\cos(\theta) + (-2A - 2B + 2A\theta)e^{-\theta}\sin(\theta)\right] + & \\ 2\left[(A - A\theta + B\theta)e^{-\theta}\cos(\theta) + (B - B\theta - A\theta)e^{-\theta}\sin(\theta)\right] & \\ +2\left[A\theta e^{-\theta}\cos(\theta) + B\theta e^{-\theta}\sin(\theta)\right] &= e^{-\theta}\cos(\theta) \\ \left(2B - \cancel{2A} - \cancel{2B\theta} + \cancel{2A} - \cancel{2A\theta} + \cancel{2B\theta} + \cancel{2A\theta}\right) e^{-\theta}\cos(\theta) & \\ + \left(-2A - \cancel{2B} + \cancel{2A\theta} + \cancel{2B} - \cancel{2B\theta} - \cancel{2A\theta} + \cancel{2Bt}\right) e^{-\theta}\sin(\theta) & \\ &= e^{-\theta}\cos(\theta) \\ 2Be^{-\theta}\cos(\theta) + (-2A)e^{-\theta}\sin(\theta) = 1e^{-\theta}\cos(\theta) + 0e^{-\theta}\sin(\theta) & \end{aligned}$$

Comparing the left-hand-side and the right-hand-side, we get $2B = 1$ and $-2A = 0$, so $\boxed{A = 0}$ and $\boxed{B = \frac{1}{2}}$, which tells us that a particular solution is:

$$y_p(\theta) = 0\theta e^{-\theta}\cos(\theta) + \frac{1}{2}\theta e^{-\theta}\sin(\theta) = \frac{1}{2}\theta e^{-\theta}\sin(\theta)$$

<u>General solution:</u> And therefore the general solution to our differential equation is:

$$y(\theta) = y_0(\theta) + y_p(\theta) = Ae^{-\theta}\cos(\theta) + Be^{-\theta}\sin(\theta) + \frac{1}{2}\theta e^{-\theta}\sin(\theta)$$

**4.5.33.** For the $\cos^3(t)$-term, use the fact that:

$$\cos^3(x) = \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x)$$

(ridiculous, I know...)

## Section 4.6: Variation of parameters

The easiest way to do the problems in this section is to look at my differential equations handout!

The formula is:

Let $y_1(t)$ and $y_2(t)$ be the solutions to the homogeneous equation, and suppose $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$. Let:

$$\widetilde{W}(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$$

And solve:

$$\widetilde{W}(t)\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

where $f(t)$ is the inhomogeneous term.

**4.6.20.** Don't freak out! Here we have $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$. Just use the variation of parameters formula with $f$ instead of the inhomogeneous term. At some point, you should get:

$$v_1'(t) = -\sin(t)f(t)$$

and

$$v_2'(t) = \cos(t)f(t)$$

Then, to get $v_1$ and $v_2$, just integrate from 0 to $t$:

$$v_1(t) = \int_0^t -\sin(s)f(s)ds$$

$$v_2(t) = \int_0^t \cos(s)f(s)ds$$

Finally, use the fact that $y_p(t) = v_1(t)\cos(t) + v_2(t)\sin(t)$, and use the formula $\sin(t)\cos(s) - \sin(s)\cos(t)$.

## Section 6.1: Basic theory of linear differential equations

**6.1.3.** First of all, make sure that the coefficient of $y'''$ is equal to 1. Then look at the domain of each term, including the inhomogeneous term (more precisely, the part of the domain which contains the initial condition 5). Then the answer is just the intersection of the domains you found!

**6.1.7.** Use the Wronskian with $x = 0$

**6.1.17.** Verify that the three functions solve the differential equations, then show they're linearly independent (by using the Wronskian at $x = 1$)

**6.1.19.** Use the fact that $y = y_p + y_0$, where $y_p$ is the given particular solution, and $y_0$ is the general solution to the homogeneous equation (which is the span of the fundamental solution set). Then use the initial conditions to solve for the constants.

**6.1.23.** For example, for $(a)$, we have:

$$L[2y_1 - y_2] = 2L[y_1] - L[y_2] = 2x\sin(x) - (x^2+1) = 2x\sin(x) - x^2 - 1$$

So $2y_1 - y_2$ solves the equation for $(a)$

**6.1.27.** Either you can use the Wronskian with $x = 0$, or use the following reasoning: If

$$a_0 + a_1 x + a_2 x^2 \cdots + a_n x^n = 0$$

This means that for **EVERY** $x$, $x$ is a zero of $a_0 + a_1 x + a_2 x^2 \cdots + a_n x^n$ (by definition of the zero function). However, this polynomial is of degree $n$, hence cannot have more than $n$ zeros unless $a_1 = a_2 = \cdots = a_n = 0$, which we want!

## Section 6.2: Homogeneous linear equations with constant coefficients

**6.2.3, 6.2.7.** The following fact might be useful:

**Rational roots theorem:** If a polynomial $p$ has a zero of the form $r = \frac{a}{b}$, then $a$ divides the constant term of $p$ and $b$ divides the leading coefficient of $p$.

This helps you 'guess' a zero of $p$. Then use long division to factor out $p$.

**6.2.15.** The reason this is written out in such a weird way is because the auxiliary polynomial is easy to figure out! Here, the auxiliary polynomial is

$(r-1)^2(r+3)(r^2+2r+5)^2$.

**6.2.20.** General solution:

The auxiliary equation is $p(r) = r^3 + 7r^2 + 14r + 8 = 0$.

To factor this, we use the rational roots theorem, which says that if $r$ is a root of the form $\frac{a}{b}$, then $a$ divides 8 (so $a = \pm 1, \pm 2, \pm 4, \pm 8$), and $b$ divides 1 (so $b = \pm 1$). This gives us the guesses $r = \pm 1, \pm 2, \pm 4, \pm 8$:

$$p(1) = 1 + 7 + 14 + 8 = 30 \neq 0$$

$$p(-1) = -1 + 7 - 14 + 8 = 0$$

**BINGO!** Therefore $r = -1$ is a root, and to factor out $p$, we use long-division:

$$
\begin{array}{r|l}
 & X^2 \quad + 6X + 8 \\
\hline
X + 1 & X^3 + 7X^2 + 14X + 8 \\
 & - X^3 \quad - X^2 \\
\hline
 & 6X^2 + 14X \\
 & - 6X^2 \quad - 6X \\
\hline
 & 8X + 8 \\
 & - 8X - 8 \\
\hline
 & 0
\end{array}
$$

Therefore $p(r) = (r+1)(r^2+6r+8) = 0$, which gives us $r = -1$, or $r^2+6r+8 = 0$, that is:

$$r = \frac{-6 \pm \sqrt{36 - 32}}{2} = \frac{-6 \pm 2}{2} = -4, -2$$

And therefore the roots of $p$ are $\boxed{r = -1, -2, -4}$, which tells us that the general solution to our differential equation is:

$$y(t) = Ae^{-t} + Be^{-2t} + Ce^{-4t}$$

Initial conditions:

Plugging in $t = 0$, we get $y(0) = A + B + C = 1$

Differentiating, we get:

$$y'(t) = -Ae^{-t} - 2Be^{-2t} - 4Ce^{-4t}$$

Plugging in $t = 0$, we get $y'(0) = -A - 2B - 4C = -3$

Differentiating again, we get:

$$y''(t) = Ae^{-t} + 4Be^{-2t} + 16Ce^{-4t}$$

Plugging in $t = 0$, we get $y''(0) = A + 4B + 16C = 13$.

So we are led to solve the system:

$$\begin{cases} A + B + C = 1 \\ -A - 2B - 4C = -3 \\ A + 4B + 16C = 13 \end{cases}$$

Which we can solve by row-reduction (yay!!!):

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -2 & -4 & -3 \\ 1 & 4 & 16 & 13 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which gives us $\boxed{A = 1, B = -1, C = 1}$, and therefore the solution of our differential equation is:

$$y(t) = e^{-t} - e^{-2t} + e^{-4t}$$

**6.2.25.** Suppose:

$$a_0 e^{rx} + a_1 x e^{rx} + \cdots + a_{m-1} x^{m-1} e^{rx} = 0$$

Now cancel out the $e^{rx}$, and you get:

$$a_0 + a_1 x + \cdots + a_{m-1} x^{m-1} = 0$$

But $1, x, x^2 \cdots, x^{m-1}$ are linearly independent, so $a_0 = a_1 = \cdots a_{m-1} = 0$, which is what we wanted!

## Section 9.1: Introduction

**9.1.7.** Let $z = y'$, then $z' = y'' = -\frac{b}{m} y' - \frac{k}{m} y = -\frac{b}{m} z - \frac{k}{m} y$, then we get:

$$\begin{cases} y' = z \\ z' = -\frac{k}{m} y - \frac{b}{m} z \end{cases}$$

**9.1.10.** Similar to **9.1.7**

**9.1.13.** Let

$$\begin{cases} x_1 = x \\ x_2 = x' = x_1' \\ x_2' = x'' = 3x' - t^2 y + \cos(t) x \\ x_3 = y \\ x_4 = y' = x_3' \\ x_5 = y'' = x_4' \\ x_5' = y''' = -y'' + tx' - y' - e^t x \end{cases}$$

Now calculate $x_i'$ and put them in a system, using the above relations.

## Section 9.4: Linear systems in normal form

**9.4.3.** This problem is easier to do than to explain.

For example, for 9.4.1:

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \mathbf{f} = \begin{bmatrix} t^2 \\ e^t \end{bmatrix}$$

Just beware of the following: If for example $y'(t)$ doesn't contain $x(t)$, then the corresponding term in the matrix $A$ is 0. Be inspired by this to solve 9.4.3.

**9.4.7.** The trick is to let $x = w'$, $y = w''$, $z = w'''$, then:

$$\begin{cases} w' = x \\ x' = y \\ y' = z \\ z' = -w + t^2 \end{cases}$$

Now rewrite this system in matrix form.

**9.4.13, 9.4.16, 9.4.19.** Use the Wronskian! The good news is that the wronskian is very easy to calculate! Just ignore any constants and put all the two or three vectors in a matrix. For example, for 9.4.17, the (pre)-Wronskian is:

$$\widetilde{W}(t) = \begin{bmatrix} e^{2t} & e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \\ 5e^{2t} & -e^{2t} & 0 \end{bmatrix}$$

And as usual, pick your favorite point $t_0$, and evaluate $det(\widetilde{W}(t_0))$. If this is nonzero, your functions are linearly independent.

**9.4.16.** The pre-Wronskian is:

$$\widetilde{W}(t) = \begin{bmatrix} \sin(t) & \sin(2t) \\ \cos(t) & \cos(2t) \end{bmatrix}$$

Evaluating this at $t = \frac{\pi}{2}$ (notice that 0 or $\pi$ don't work), we get:

$$\widetilde{W}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

And therefore the Wronskian at $\frac{\pi}{2}$ is:

$$W\left(\frac{\pi}{2}\right) = \det \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -1$$

Since the Wronskian is nonzero at some point (here $\frac{\pi}{2}$), the two vector functions are $\boxed{\text{linearly independent}}$ on $\mathbb{R}$

**9.4.23.** Just show that the three vectors are linearly independent. To find $A$, for every vector $\mathbf{x}$ given, calculate $\mathbf{x}'(t)$ for every vector $\mathbf{x}$ and just write this in terms of $\mathbf{x}(t)$. This gives the first, second, and third column of $A$ respectively.

**9.4.27.** Linear operator (in this case) is just another word for linear transformation. Just show that $L[\mathbf{x}+\mathbf{y}] = L[\mathbf{x}] + L[\mathbf{y}]$ and that $L[c\mathbf{x}] = cL[\mathbf{x}]$

## Section 9.5: Homogeneous linear systems with constant coefficients

If you're lost about this, check out the handout 'Systems of differential equations' on my website! Essentially all you have to do is to find the eigenvalues and eigenvectors of $A$.

Also, to deal with the 'finding the eigenvalues' part, remember the following theorem:
**Rational roots theorem:** If a polynomial $p$ has a zero of the form $r = \frac{a}{b}$, then $a$ divides the constant term of $p$ and $b$ divides the leading coefficient of $p$.

This helps you 'guess' a zero of $p$. Then use long division to factor out $p$.

**9.5.17.** First, draw two lines, one spanned by $\mathbf{u_1}$ and the other one spanned by $\mathbf{u_2}$. Then on the first line, draw arrows pointing *away* from the origin (because of the $e^{2t}$-term in the solution, points on that line *move away* from the origin). On the second line, draw arrows pointing *towards* the origin (because of the $e^{-2t}$-term, solutions move towards the origin). Finally, for all the other points, all you have to do is to 'connect' the arrows (think of it like drawing a force field or a velocity field).

If you want a picture of how the answer looks like, google 'saddle phase portrait differential equations' and under images, check out the second image you get!

**9.5.21.** The fundamental solution set is just the matrix whose columns are the solutions to your differential equation. Basically find the general solution to your differential equation, ignore the constants, and put everything else in a matrix!

**9.5.31.** Eigenvalues:

$$\det(\lambda I - A) = (\lambda - 1)^2 - 9 = 0 \Rightarrow \lambda - 1 = \pm 3 \Rightarrow \lambda = -2, 4$$

Eigenvectors:

$$Nul(-2I - A) = Nul \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} = Nul \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = Span \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$Nul(4I - A) = Nul \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = Nul \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = Span \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Therefore, the general solution to our differential equation is:

$$\mathbf{x}(t) = Ae^{-2t}\begin{bmatrix}1\\-1\end{bmatrix} + Be^{4t}\begin{bmatrix}1\\1\end{bmatrix}$$

Now plug in $t = 0$:

$$\mathbf{x}(0) = A\begin{bmatrix}1\\-1\end{bmatrix} + B\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

which leads us to solve the system:

$$\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

Row-reducing, we get:

$$\begin{bmatrix}1 & 1 & 3\\-1 & 1 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\end{bmatrix}$$

which gives us $A = 1$, $B = 2$, and therefore our solution is:

$$\mathbf{x}(t) = e^{-2t}\begin{bmatrix}1\\-1\end{bmatrix} + 2e^{4t}\begin{bmatrix}1\\1\end{bmatrix}$$

**9.5.35.** For $(c)$, you don't need to derive the relations, just solve the following equation for $\mathbf{u_2}$: $A\mathbf{u_2} = \mathbf{u_1}$.

## Section 9.6: Complex eigenvalues

Again, for all those problems, look at the handout 'Systems of differential equations', where everything is discussed in more detail!

**9.6.19.** Use use equation (10) on page 541 with $m_1 = m_2 = 1$, $k_1 = k_2 = k_3 = 2$.

**Note:** The trick where you let $y_1 = x_1$, $y_2 = x_1'$, $y_3 = x_2$, $y_4 = x_2'$ is important to remember! It allows you to convert second-order differential equations into a system of differential equations!

## Section 9.7: Nonhomogeneous linear equations

Again, the handout 'Systems of differential equations' goes through this in more detail!

**Note:** In what follows, $\mathbf{a} = \begin{bmatrix}a_1\\a_2\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ are $2-$vectors.

**9.7.3.** Guess $\mathbf{f}(t) = e^t\mathbf{a}$

**9.7.9.** Guess $\mathbf{f}(t) = e^{2t}\mathbf{a} + \cos(t)\mathbf{b} + \sin(t)\mathbf{c} + \mathbf{d} + \mathbf{e}t$

**9.7.13, 9.7.15.** The formula is:

$$\left(\widetilde{W}(t)\right)\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \mathbf{f}$$

where $\widetilde{W}(t)$ is the (pre)-Wronskian, or fundamental matrix for your system (essentially the solutions but without the constants).

## Section 10.2: Method of separation of variables

**10.2.1, 10.2.3, 10.2.5.** Just solve your equation the way you would usually do (for 5, use undetermined coefficients) and plug in the initial conditions. You may or may not find a contradiction! If you find $0 = 0$, that usually means there are infinitely many solutions, depending on your constant $A$ or $B$.

**10.2.9, 10.2.12.** You have to split up your analysis into three cases:

**Case 1:** $\lambda > 0$. Then let $\lambda = \omega^2$, where $\omega > 0$. This helps you get rid of square roots.

**Case 2:** $\lambda = 0$.

**Case 3:** $\lambda < 0$. Then $\lambda = -\omega^2$, where $\omega < 0$.

In each case, solve the equation and plug in your initial condition. You may or may not get a contradiction. Also, remember that $y$ has to be nonzero!

**10.2.21, 10.2.23.** Follow the outline given in the sections 'Heat equation' and 'Wave equation' in my Partial Differential Equations-Handouts. You don't need to worry about Fourier series, as you can just compare the coefficients.

**10.2.21.**

**Step 1: Separation of variables.** Suppose:

$$u(x,t) = X(x)T(t) \tag{1}$$

Plug (1) into the differential equation we get:

$$\begin{aligned} \left(X(x)T(t)\right)_{tt} &= 9\left(X(x)T(t)\right)_{xx} \\ X(x)T''(t) &= 9X''(x)T(t) \end{aligned}$$

Rearrange and get:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{9T(t)} \tag{2}$$

Now $\frac{X''(x)}{X(x)}$ *only* depends on $x$, but by (2) *only* depends on $t$, hence it is constant:

$$\begin{aligned} \frac{X''(x)}{X(x)} =& \lambda \\ X''(x) =& \lambda X(x) \end{aligned} \tag{3}$$

Also, we get:

$$\begin{aligned} \frac{T''(t)}{9T(t)} =& \lambda \\ T''(t) =& 9\lambda T(t) \end{aligned} \tag{4}$$

**Step 2:** Consider (3):

$$X''(x) = \lambda X(x)$$

Now use the **boundary conditions**:

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u(\pi,t) = X(\pi)T(t) = 0 \Rightarrow X(\pi)T(t) = 0 \Rightarrow X(\pi) = 0$$

Hence we get:

$$\begin{cases} X''(x) =& \lambda X(x) \\ X(0) =& 0 \\ X(\pi) =& 0 \end{cases} \tag{5}$$

**Step 3: Eigenvalues/Eigenfunctions.** The auxiliary polynomial of (5) is $p(\lambda) = r^2 - \lambda$

Now we need to consider 3 cases:

<u>Case 1:</u> $\lambda > 0$, then $\lambda = \omega^2$, where $\omega > 0$

Then:

$$r^2 - \lambda = 0 \Rightarrow r^2 - \omega^2 = 0 \Rightarrow r = \pm\omega$$

Therefore:

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$

Now use $X(0) = 0$ and $X(\pi) = 0$:

$$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A \Rightarrow X(x) = Ae^{\omega x} - Ae^{-\omega x}$$

$X(\pi) = 0 \Rightarrow Ae^{\omega\pi} - Ae^{-\omega\pi} = 0 \Rightarrow Ae^{\omega\pi} = Ae^{-\omega\pi} \Rightarrow e^{\omega\pi} = e^{-\omega\pi} \Rightarrow \omega\pi = -\omega\pi \Rightarrow \omega = 0$

But this is a **contradiction**, as we want $\omega > 0$.

Case 2: $\lambda = 0$, then $r = 0$, and:

$$X(x) = Ae^{0x} + Bxe^{0x} = A + Bx$$

And:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = Bx$$

$$X(\pi) = 0 \Rightarrow B = 0 \Rightarrow X(x) = 0$$

Again, a **contradiction** (we want $X \not\equiv 0$, because otherwise $u(x,t) \equiv 0$)

Case 3: $\lambda < 0$, then $\lambda = -\omega^2$, and:

$$r^2 - \lambda = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm\omega i$$

Which gives:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

Again, using $X(0) = 0$, $X(\pi) = 0$, we get:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B\sin(\omega x)$$

$$X(\pi) = 0 \Rightarrow B\sin(\omega\pi) = 0 \Rightarrow \sin(\omega\pi) = 0 \Rightarrow \omega = m, \qquad (m = 1, 2, \cdots)$$

This tells us that:

$$\text{Eigenvalues:} \lambda = -\omega^2 = -m^2 \qquad (m = 1, 2, \cdots)$$
$$\text{Eigenfunctions:} X(x) = \sin(\omega x) = \sin(mx) \tag{6}$$

**Step 4:** Deal with (4), and remember that $\lambda = -m^2$:

$$T''(t) = 3\lambda T(t)$$

Aux: $r^2 = -9m^2 \Rightarrow r = \pm 3mi \qquad (m = 1, 2, \cdots)$

$$T(t) = A_m\cos(3mt) + B_m\sin(3mt)$$

**Step 5:** Take linear combinations:

$$u(x,t) = \sum_{m=1}^{\infty} T(t)X(x) = \sum_{m=1}^{\infty} \left(\widetilde{A_m}\cos(3mt) + \widetilde{B_m}\sin(3mt)\right)\sin(mx) \tag{7}$$

**Step 6:** Use the initial condition $u(x,0) = 6\sin(2x) + 2\sin(6x)$:

Plug in $t = 0$ in (7), and you get:

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx) = 6\sin(2x) + 2\sin(6x) \qquad \text{on}(0,\pi) \tag{8}$$

Equating coefficients, you get:

$$\begin{aligned} A_2 &= 6 && \text{(coefficient of } \sin(2x)\text{)} \\ A_6 &= 2 && \text{(coefficient of } \sin(6x)\text{)} \\ A_m &= 0 && \text{(for all other } m\text{)} \end{aligned}$$

**Step 7:** Use the initial condition: $\frac{\partial u}{\partial t}(x,0) = 11\sin(9x) - 14\sin(15x)$:
First differentiate (7) with respect to $t$:

$$\frac{\partial u}{\partial t}(x,t) = \sum_{m=1}^{\infty} \left(-3mA_m \sin(3mt) + 3mB_m \cos(3mt)\right)\sin(mx) \tag{9}$$

Now plug in $t = 0$ in (9):

$$\frac{\partial u}{\partial t}(x,0) = \sum_{m=1}^{\infty} 3m\widetilde{B_m} \sin(mx) = 11\sin(9x) - 14\sin(15x) \tag{10}$$

Equating coefficients, you get:

$$\begin{aligned} 27B_9 &= 11 && \text{(coefficient of } \sin(9x)\text{)} \\ 45B_{15} &= -14 && \text{(coefficient of } \sin(15x)\text{)} \\ B_m &= 0 && \text{(for all other } m\text{)} \end{aligned}$$

That is:

$$\begin{aligned} B_9 &= \frac{11}{27} && \text{(coefficient of } \sin(9x)\text{)} \\ B_{15} &= -\frac{14}{45} && \text{(coefficient of } \sin(15x)\text{)} \\ B_m &= 0 && \text{(for all other } m\text{)} \end{aligned}$$

**Step 8:** Conclude using (7) and the coefficients $A_m$ and $B_m$ you found:

$$u(x,t) = \sum_{m=1}^{\infty} \left(A_m \cos(3mt) + B_m \sin(3mt)\right)\sin(mx) \tag{11}$$

where:

$$\begin{aligned} A_2 &= 6 \\ A_6 &= 2 \\ A_m &= 0 \qquad \text{(for all other } m) \end{aligned}$$

and

$$\begin{aligned} B_9 &= \frac{11}{27} \\ B_{15} &= -\frac{14}{45} \\ B_m &= 0 \qquad \text{(for all other } m) \end{aligned}$$

**Note:** In this *special* case, you can write $u(x,t)$ in the following nice form:

(12)
$$u(x,t) = 6\cos(6t)\sin(2x) + 2\cos(18t)\sin(6x) + \frac{11}{27}\sin(27t)\sin(9x) - \frac{14}{45}\sin(45t)\sin(15x)$$

## Section 10.3: Fourier series

**10.3.1, 10.3.5.** $f$ is even if $f(-x) = f(x)$, $f$ is odd if $f(-x) = -f(x)$.

**10.3.7.** Just calculate $fg(-x) = f(-x)g(-x)$

For what follows, use the following formulas:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right) \right\}$$

$$a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos\left(\frac{n\pi x}{T}\right) dx$$

$$b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin\left(\frac{n\pi x}{T}\right) dx$$

Where $T$ is such that $f$ is defined on $(-T, T)$

**10.3.11.**

$$\begin{aligned}A_0 =&\frac{\int_{-2}^{2} f(x)1dx}{\int_{-2}^{2} 1^2 dx}\\=&\frac{1}{4}\left(\int_{-2}^{0} 1dx + \int_0^2 xdx\right)\\=&\frac{1}{4}\left(2+\left[\frac{x^2}{2}\right]_0^2\right)\\=&\frac{1}{4}(2+2)\\=&1\end{aligned}$$

$$\begin{aligned}A_m =&\frac{\int_{-2}^{2} f(x)\cos\left(\frac{\pi m x}{2}\right)dx}{\int_{-2}^{2}\cos^2\left(\frac{\pi m x}{2}\right)dx}\\=&\frac{1}{2}\left(\int_{-2}^{0}\cos\left(\frac{\pi m x}{2}\right)dx + \int_0^2 x\cos\left(\frac{\pi m x}{2}\right)dx\right)\\=&\frac{1}{2}\left(\left[\frac{2}{\pi m}\sin\left(\frac{\pi m x}{2}\right)\right]_{-2}^{0} + \left[\frac{2}{\pi m}x\sin\left(\frac{\pi m x}{2}\right)\right]_0^2 - \int_0^2 \frac{2}{\pi m}\sin\left(\frac{\pi m x}{2}\right)dx\right)\\=&\frac{1}{2}\left(0+0-\frac{2}{\pi m}\left[-\frac{2}{\pi m}\cos\left(\frac{\pi m x}{2}\right)\right]_0^2\right)\\=&\frac{2}{(\pi m)^2}\left(\cos(\pi m)-\cos(0)\right)\\=&\frac{2}{(\pi m)^2}\left((-1)^m-1\right)\end{aligned}$$

$B_0 = 0$ by convention.

$$\begin{aligned}B_m =&\frac{\int_{-2}^{2} f(x)\sin\left(\frac{\pi m x}{2}\right)dx}{\int_{-2}^{2}\sin^2\left(\frac{\pi m x}{2}\right)dx}\\=&\frac{1}{2}\left(\int_{-2}^{0}\sin\left(\frac{\pi m x}{2}\right)dx + \int_0^2 x\sin\left(\frac{\pi m x}{2}\right)dx\right)\\=&\frac{1}{2}\left(\left[-\frac{2}{\pi m}\cos\left(\frac{\pi m x}{2}\right)\right]_{-2}^{0} + \left[-\frac{2}{\pi m}x\cos\left(\frac{\pi m x}{2}\right)\right]_0^2 + \int_0^2 \frac{2}{\pi m}\cos\left(\frac{\pi m x}{2}\right)dx\right)\\=&\frac{1}{2}\left(-\frac{2}{\pi m}+\frac{2}{\pi m}\cos(\pi m)-\frac{4}{\pi m}\cos(\pi m)+0+\left[\frac{4}{(\pi m)^2}\sin\left(\frac{\pi m x}{2}\right)\right]_0^2\right)\\=&\frac{1}{2}\left(-\frac{2}{\pi m}+\frac{2}{\pi m}(-1)^m-\frac{4}{\pi m}(-1)^m+0-0\right)\\=&-\frac{1}{\pi m}-\frac{(-1)^m}{\pi m}\\=&\frac{1}{\pi m}\left(-1+(-1)^{m+1}\right)\end{aligned}$$

It follows that:

$$f(x)\text{``}=\text{''}1+\sum_{m=1}^{\infty}\left(\frac{2}{(\pi m)^2}\left((-1)^m-1\right)\cos\left(\frac{\pi m x}{2}\right)+\frac{1}{\pi m}\left(-1+(-1)^{m+1}\right)\sin\left(\frac{\pi m x}{2}\right)\right)$$

**10.3.17, 10.3.19.** The Fourier series converges to $f(x)$ if $f$ is **continuous** at $x$, and converges to $\frac{f(x^+)+f(x^-)}{2}$ if $f$ is **discontinuous** at $x$. As for the endpoints $T$ and $-T$, the fourier series converges to the average of $f$ at those endpoints.

**10.3.26.** Just show:

$$\int_{-1}^{1}\cos\left(\frac{(2m-1)\pi}{2}x\right)\sin\left(\frac{(2n-1)\pi}{2}x\right)dx=0$$
$$\int_{-1}^{1}\cos\left(\frac{(2m-1)\pi}{2}x\right)\cos\left(\frac{(2n-1)\pi}{2}x\right)dx=0$$
$$\int_{-1}^{1}\sin\left(\frac{(2m-1)\pi}{2}x\right)\sin\left(\frac{(2n-1)\pi}{2}x\right)dx=0$$

for all $m$ and $n$.

Use the following formulas:

$$2\cos(A)\cos(B)=\cos(A+B)+\cos(A-B), 2\sin(A)\sin(B)=\cos(A-B)-\cos(A+B)$$

as well as the fact that odd-ness (for the first one).

**10.3.27.** Just calculate:

$$\frac{\int_{-1}^{1}f(x)g(x)dx}{\int_{-1}^{1}g(x)^2dx}$$

for every function $g(x)$ in 10.3.27 (this follows from formula (20) on page 588).

## Section 10.4: Fourier cosine and sine series

**IMPORTANT NOTE:** The book uses the following trick **A LOT**:

Namely, suppose that when you calculate your coefficients $A_m$ or $B_m$, you get something like: $A_m=\frac{(-1)^{m+1}+1}{\pi m}$.

Then notice the following: If $m$ is even, then $(-1)^{m+1}+1=0$, so $A_m=0$, and if $m$ is odd, $(-1)^{m+1}+1=-2$, and $A_m=\frac{-2}{\pi m}$.

So at some point, you would like to say:

$$f(x) “=” \sum_{m=1, modd}^{\infty} A_m \cos(mx)$$

The way you do this is as follows: Since $m$ is odd $m = 2k-1$, for $k = 1, 2, 3 \cdots$, and so the sum becomes:

$$f(x) “=” \sum_{k=1}^{\infty} \frac{-2}{\pi(2k-1)} \cos((2k-1)x)$$

**10.4.1, 10.4.3.** $\pi$-periodic extension just means ‘repeat the graph of $f$’.

The even-$2\pi$ periodic extension is just the function:

$$f_e(x) = \begin{cases} f(-x) & \text{if} -\pi < x < 0 \\ f(x) & \text{if} 0 < x < \pi \end{cases}$$

The odd-$2\pi$ periodic extension is just the function:

$$f_o(x) = \begin{cases} -f(-x) & \text{if} -\pi < x < 0 \\ 0 & \text{if} x = 0 \\ f(x) & \text{if} 0 < x < \pi \end{cases}$$

And repeat all those graphs!

**10.4.5, 10.4.7, 10.4.9.** Use the formulas:

$$f(x) “=” \sum_{m=0}^{\infty} A_m \cos\left(\frac{\pi m x}{T}\right)$$

where:

$$A_0 = \frac{1}{T} \int_0^T f(x) dx$$

$$A_m = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{\pi m x}{T}\right) dx$$

**10.4.11, 10.4.13.** Use the formulas:

$$f(x) “=” \sum_{m=0}^{\infty} B_m \sin\left(\frac{\pi m x}{T}\right)$$

where:

$$B_0 = 0$$

$$B_m = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{\pi m x}{T}\right) dx$$

**10.4.17, 10.4.19.** See next section!

## Section 10.5: The heat equation

The best advice I can give you is: Read the PDE handout, specifically the section about the heat equation! It outlines all the important steps you'll need!

Also read the **important note** I wrote in the previous section!

**10.5.7.** Imitate Example 2! Your solution is

$$u(x,t) = v(x) + w(x,t)$$

where $v(x) = 5 + \frac{5x}{\pi}$ and $w(x,t)$ solves the corresponding homogeneous equation with $w(0,t) = 0, w(\pi,t) = 0$ but with $w(x,0) = \sin(3x) - \sin(5x) - v(x)$.

**10.5.9.** Don't worry about this for the exam, but basically because we're dealing with an inhomogeneous solution, the general solution $u(x,t)$ is of the following form:

$$u(x,t) = v(x) + w(x,t)$$

where $v(x)$ is a **particular** solution to the differential equation, and $w(x,t)$ is the general solution to the **homogeneous** equation (36), (37), (38) on page 671 (careful about the initial term, it's $w(x,0) = f(x) - v(x)$, not $w(x,0) = f(x)$)

To find $v$ use formula (35) on page 671, and to find $w$, solve equations (36), (37), (38).

**10.5.15, 10.5.17. Note:** This is just an outline. On your homework, please fill in all the details.

This time assume $u(x,y,t) = X(x)Y(y)T(t)$. Plugging $u$ into the PDE we get:

$$X(x)Y(y)T'(t) = X''(x)Y(y)T(t) + XY''(y)T(t)$$

And dividing by $X(x)Y(t)T(t)$, we get:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}$$

Now the right-hand-side depends only on $x$, and $y$, but also only on $t$ (by the left-hand-side), hence it is constant, which gives us:

$$\frac{T'(t)}{T(t)} = \lambda = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}$$

So in particular $T'(t) = \lambda T(t)$, and also:

$$\frac{X''(x)}{X(x)} = \lambda - \frac{Y''(y)}{Y(y)}$$

But now notice that the left-hand-side depends only on $x$, and only on $y$ (by the right-hand-side), hence it is also constant, and we get:

$$\frac{X''(x)}{X(x)} = \mu = \lambda - \frac{Y''(y)}{Y(y)}$$

So in particular $X''(x) = \mu X(x)$. Now use the boundary condition $X'(0) = X'(\pi) = 0$ and cases to argue that $\mu = m^2$ for $m = 0, 1, \cdots$, and $X_m = A_m \cos(mx)$ which gives:

$$\frac{Y''(y)}{Y(y)} = \lambda - \mu = \lambda - m^2$$

But now use the boundary condition $Y(0) = Y(\pi) = 0$ and cases to argue that $\lambda - m^2 = n^2$ for $n = 1, 2, 3, \cdots$, (and so $\lambda = m^2 + n^2$), and $Y_n = B_n \sin(ny)$.

Finally, using $T'(t) = \lambda T = (m^2+n^2)T$, we get: $T(t) = e^{(m^2+n^2)t}$, and we finally obtain:

$$u_{mn}(x, y, t) = X(x)Y(y)T(t) = A_m B_n \cos(mx)\sin(ny)e^{(m^2+n^2)t}$$

And finally the general solution is:

$$u(x, y, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{mn}\cos(mx)\sin(ny)e^{(m^2+n^2)t}$$

Finally, all you have to do is to plug in $t = 0$ to get:

$$u(x, y, 0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{mn}\cos(mn)\sin(ny)$$

For **10.5.15**, you just have to compare terms and get $C_{6,4} = 1$ and $C_{1,11} = -3$ and everything else $= 0$, which tells you:

$$u(x, y, t) = \cos(6x)\sin(4y)e^{52t} - 3\cos(x)\sin(11y)e^{122t}$$

And for **10.5.17**, you use 'hugging' (or orthogonality) to get:

$$C_{0n} = \frac{\int_0^\pi \int_0^\pi y(1)\sin(ny)dxdy}{\int_0^\pi \int_0^\pi 1^2 \sin^2(ny)dxdy} = \frac{\pi \int_0^\pi y \sin(ny)dy}{\frac{\pi}{2}} = 2\frac{(-1)^{n+1}}{n}$$

If $m \geq 1$

$$C_{mn} = \frac{\int_0^\pi \int_0^\pi y\cos(mx)\sin(ny)dxdy}{\int_0^\pi \int_0^\pi \cos^2(mx)\sin^2(ny)dxdy} = \frac{\left(\int_0^\pi \cos(mx)dx\right)\left(\int_0^\pi y\sin(ny)dy\right)}{\frac{1}{4}} = 0$$

Which ultimately gives us:

$$u(x, y, t) = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n}e^{-n^2 t}\sin(ny)$$

## Section 10.6: The wave equation

Read the PDE handout, specifically the section about the wave equation! It outlines all the important steps you'll need!

## Section 10.7: Laplace's equation

Read the PDE handout, specifically the section about Laplace's equation! It outlines all the important steps you'll need!

The most important thing to remember is that when you solve for $Y(y)$, your solution might involve exponentials, i.e.

$$Y(y) = Ae^{wy} + Be^{-wy}$$

for some constants $A, B, w$ (which might depend on $w$). Do **NOT** use this form! Instead, use the fact that:

$$\frac{e^w + e^{-w}}{2} = \cosh(w)$$
$$\frac{e^w - e^{-w}}{2} = \sinh(w)$$

and write:

$$Y(y) = A\cosh(wy) + B\sinh(wy)$$

where $A$ and $B$ might be constants different from above (but still call them $A$ and $B$).

This will simplify your algebra by a **LOT**, trust me!