Math 54 - Practice Final Exam Solutions
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Disclaimer: These solutions should be correct, but I may have made some typos. If unsure about a solution, ask ;)

1. (a) Find the inverse of the following matrix.
$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & -2 \\ -3 & 12 & -32 \end{pmatrix}$$

(b) Solve the following system using the inverse found above.
$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & -2 \\ -3 & 12 & -32 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$$

Solution. (a) We use Gaussian elimination as follows:
$$\left(\begin{array}{ccc|ccc} 1 & -2 & 4 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ -3 & 12 & -32 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & -2 & 4 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -2 & 6 & -3 & 1 \end{array}\right)$$
$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 4 & -1 \\ 0 & 2 & 0 & -19 & 10 & -3 \\ 0 & 0 & -2 & 6 & -3 & 1 \end{array}\right).$$

Dividing the second row through by 2 and the third by -2, we conclude that the inverse of the given matrix is
$$\begin{pmatrix} -6 & 4 & -1 \\ -19/2 & 5 & -3/2 \\ -3 & 3/2 & -1/2 \end{pmatrix}.$$

(b) Since the coefficient matrix is invertible, the solution is
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & -2 \\ -3 & 12 & -32 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 & 4 & -1 \\ -19/2 & 5 & -3/2 \\ -3 & 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}.$$

□

2. Let W be the set of all invertible $n \times n$ matrices. Is W a subspace of M_{nn}?

Solution. No, W is not a subspace of M_{nn}. The simplest reason why is because the zero matrix is not invertible and so is not in W. But also, W is not closed under addition — for example, the identity matrix I is in W, as is its negative $-I$, but their sum $I + (-I) = 0$ is not in W. □

3. Determine if the polynomials $x + 1$, $2x^2 + 3$, and $3x - 5$ are linearly independent. Do they span P_2?

Solution. We can use the coordinate vectors of these polynomials as the columns of a matrix

$$\begin{pmatrix} 1 & 3 & -5 \\ 1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}.$$

Now, these polynomials are linearly indepedent if and only if the columns of this matrix are linearly indepdent, which in turn is true if and only if the echelon form of the matrix has a pivot in each column. We have

$$\begin{pmatrix} 1 & 3 & -5 \\ 1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -5 \\ 0 & -3 & 8 \\ 0 & 0 & 16 \end{pmatrix}$$

so we conclude that the columns, and thus the polynomials are linearly independent.

Since we have 3 linearly independent vectors and P_2 is 3-dimensional, they must span P_2 (which we can also see from the echelon form above). □

4. Find bases for the null space, row space, and column space of the following matrix.

$$\begin{pmatrix} 1 & 1 & -3 & 3 & 5 \\ 4 & 4 & -14 & 12 & 22 \\ -3 & -3 & 5 & -6 & -5 \\ 1 & 1 & 3 & 6 & 5 \end{pmatrix}$$

Solution. The echelon form of this matrix is

$$\begin{pmatrix} 1 & 1 & -3 & 3 & 5 \\ 4 & 4 & -14 & 12 & 22 \\ -3 & -3 & 5 & -6 & -5 \\ 1 & 1 & 3 & 6 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -3 & 3 & 5 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this we know that basis for the row space is

$$\begin{pmatrix} 1 & 1 & -3 & 3 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -2 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 3 & 6 \end{pmatrix}$$

i.e. the nonzero rows of the echelon form. Also, a basis for the column space is

$$\begin{pmatrix} 1 \\ 4 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -14 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 12 \\ -6 \\ 6 \end{pmatrix},$$

i.e. the columns of the matrix which correspond to pivot columns of the echelon form. Finally, an arbitrary vector in the null space has the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -s+4t \\ s \\ t \\ -2t \\ t \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

Hence a basis for the null space is

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

$\square$

5. (a) Let $X = \left(\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix}\right)$. Let $T : M_{22} \to M_{22}$ be the function defined by $T(A) = AX - XA$. Prove that T is a linear transformation.

(b) Find a basis for the null space of T. (The null space of T is defined as the set of all matrices A such that $T(A) = 0$)

Solution. (a) Let $A, B \in M_{22}$. Then

$$\begin{aligned} T(A + B) &= (A + B)X - X(A + B) \\ &= AX + BX - XA - XB \\ &= (AX - XA) + (BX - XB) \\ &= T(A) + T(B). \end{aligned}$$

Also, if r is a scalar, then

$$\begin{aligned} T(rA) &= (rA)X - X(rA) \\ &= rAX - rXA \\ &= r(AX - XA) \\ &= rT(A) \end{aligned}$$

so T satisfies the two required properties of being a linear transformation.

(b) Let A be a 2×2 matrix such that $T(A) = 0$. We first determine what A looks like. Suppose that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Since $T(A) = AX - XA = 0$, we must have $AX = XA$. Computing both sides we have

$$AX = XA$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\begin{pmatrix} a & 2a - b \\ c & 2c - d \end{pmatrix} = \begin{pmatrix} a + 2c & b + 2d \\ -c & -d \end{pmatrix}.$$

Now, comparing corresponding entries, we see that c must be 0 and $2a - b = b + 2d$, so $a = b + d$. Thus the matrix A has the form

$$\begin{pmatrix} b + d & b \\ 0 & d \end{pmatrix} = b \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence a basis for the null space of T is

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\square$

6. Let V be an inner product space. Using the Cauchy-Schwartz inequality, prove that for any vectors $\mathbf{u}, \mathbf{v} \in V$, $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Solution. By the Cauchy-Schartz inequality, we know that $\mathbf{u} \cdot \mathbf{v} \leq \|u\|\|v\|$. Thus

$$
\begin{aligned}
\|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\
&= \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\
&\leq \mathbf{u} \cdot \mathbf{u} + 2\|\mathbf{u}\|\|\mathbf{v}\| + \mathbf{v} \cdot \mathbf{v} \\
&= \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|^2 \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.
\end{aligned}
$$

Taking the square root of both sides we get the desired inequality. □

7. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an orthonormal basis for an inner product space V. Prove that the coordinate vector relative to this basis of a vector $\mathbf{x} \in V$ is

$$
[\mathbf{x}] = \begin{pmatrix} \mathbf{x} \cdot \mathbf{v}_1 \\ \vdots \\ \mathbf{x} \cdot \mathbf{v}_n \end{pmatrix}.
$$

Solution. Suppose that $\mathbf{x} = c_1 \mathbf{v}_1 + \ldots + c_n \mathbf{v}_n$. Then, taking the inner product of both sides with $\mathbf{v}_i$ we have

$$
\begin{aligned}
\mathbf{x} \cdot \mathbf{v}_i &= (c_1 \mathbf{v}_1 + \ldots + c_n \mathbf{v}_n) \cdot \mathbf{v}_i \\
&= c_1 \mathbf{v}_1 \cdot \mathbf{v}_i + \ldots + c_i \mathbf{v}_i \cdot \mathbf{v}_i + \ldots + c_n \mathbf{v}_n \cdot \mathbf{v}_i.
\end{aligned}
$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are orthonormal, each inner product on the right side is 0 except for $\mathbf{v}_i \cdot \mathbf{v}_i$, which is 1. Thus the only nonzero term on the right side is $c_i \mathbf{v}_i \cdot \mathbf{v}_i = c_i$ and so we conclude that

$$
c_i = \mathbf{x} \cdot \mathbf{v}_i.
$$

Hence the coordinate vector of $\mathbf{x}$ is as claimed. □

8. (a) Find the determinant of the following matrix.

$$
\begin{pmatrix} -2 & 3 & 2 \\ -1 & 3 & 0 \\ 4 & -3 & 1 \end{pmatrix}
$$

(b) Is the above matrix invertible?

Solution. (a) Using a cofactor expansion along the second row, we have

$$
\begin{vmatrix} -2 & 3 & 2 \\ -1 & 3 & 0 \\ 4 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ -3 & 1 \end{vmatrix} + 3 \begin{vmatrix} -2 & 2 \\ 4 & 1 \end{vmatrix} = 9 + 3(-10) = -21.
$$

(b) Since the determinant is not 0, this matrix is invertible. □

9. Find bases for the eigenspaces of the following matrix.

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

Solution. The characteristic polynomial of this matrix is (using a cofactor expansion along the first column)

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -2 & -1 \\ -1 & \lambda - 3 & -1 \\ -1 & -2 & \lambda - 2 \end{vmatrix}$$

$$= (\lambda - 2)\begin{vmatrix} \lambda - 3 & -1 \\ -2 & \lambda - 2 \end{vmatrix} + \begin{vmatrix} -2 & -1 \\ -2 & \lambda - 2 \end{vmatrix} - \begin{vmatrix} -2 & -1 \\ \lambda - 3 & -1 \end{vmatrix}$$

$$= (\lambda - 2)(\lambda^2 - 5\lambda + 4) - (-2\lambda + 2) - (\lambda - 1)$$

$$= (\lambda - 2)(\lambda - 4)(\lambda - 1) - 2(\lambda - 1) - (\lambda - 1)$$

$$= (\lambda - 1)[(\lambda - 2)(\lambda - 4) - 2 - 1]$$

$$= (\lambda - 1)(\lambda^2 - 6\lambda + 5)$$

$$= (\lambda - 1)^2(\lambda - 5).$$

Thus the eigenvalues are 5 and 1. Computing a basis for $\mathrm{NS}(\lambda I - A)$ for each eigenvalue λ, we find that a bases for W_5 and W_1 respectively are

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

□

10. Determine if the following matrices are diagonalizable. If so, diagonalize them.

$$(a)\ \begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Solution. (a) The eigenvalues of this matrix are 5 and 4, and bases for W_5 and W_4 respectively are

$$\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$$

Since we have three linearly indepedent eigenvectors, this matrix is diagonalizable, and we can diagonalize it as

$$\begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}^{-1}.$$

(b) The eigenvalues of this matrix are 4 and 5, and bases for W_4 and W_5 respectively are

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since there are only two linearly independent eigenvectors, this matrix is not diagonalizable. $\square$

11. (a) Prove that if ξe^{rt} is a solution of a system of differential equations $\mathbf{x}' = A\mathbf{x}$, then r is an eigenvalue of A and ξ is an associated eigenvector.

(b) Solve the following system and draw its phase portrait.

$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

Solution. (a) For $\mathbf{x} = \xi e^{rt}$ to be a solution of $\mathbf{x}' = A\mathbf{x}$, we must have

$$(\xi e^{rt})' = A(\xi e^{rt})$$
$$r\xi e^{rt} = e^{rt} A\xi$$
$$r\xi = A\xi.$$

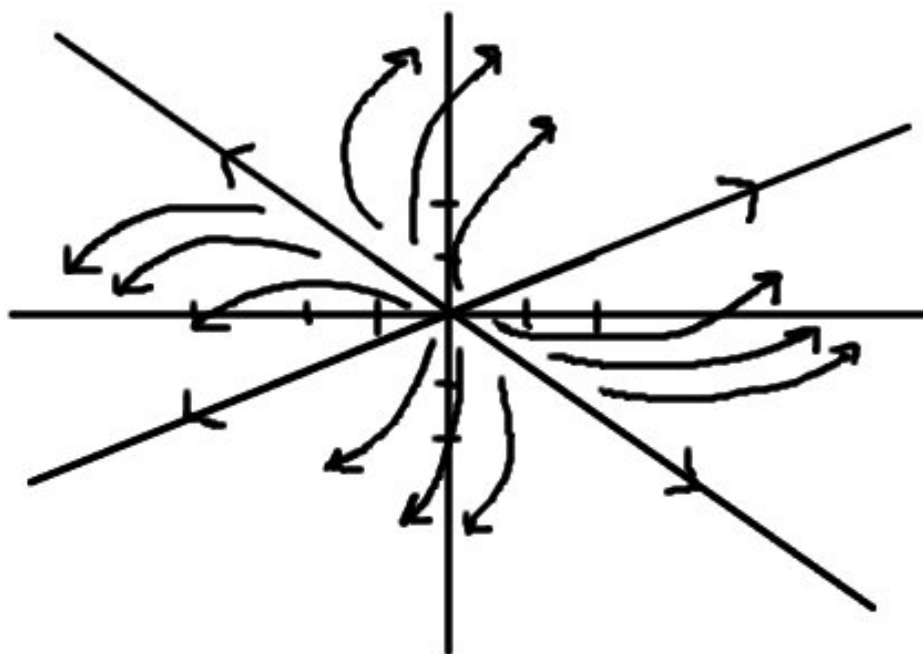Hence r must be an eigenvalue of A and ξ must be an associated eigenvector.

(b) The eigenvalues of A are 5 and 2, and eigenvectors corresponding to each of these respectively are

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence the general solution of the system is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}.$$

The phase portrait is

12. Consider the following system of linear differential equations.

$$\mathbf{x}' = \begin{pmatrix} 6 & 5 \\ 2 & -3 \end{pmatrix} \mathbf{x}$$

(a) Find the special fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = I$.

(b) Solve the following initial value problem using the fundamental matrix found in (a).

$$\mathbf{x}' = \begin{pmatrix} 6 & 5 \\ 2 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

(c) Draw the phase portrait of the given system.

Solution. (a) The eigenvalues of A are 7 and -4, and eigenvectors corresponding to these respectively are

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Hence two linearly independent solutions of the system are

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} e^{7t} \text{ and } \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-4t}.$$

From this we can form a fundamental matrix

$$\Psi(t) = \begin{pmatrix} 5e^{7t} & -e^{-4t} \\ e^{7t} & 2e^{-4t} \end{pmatrix}.$$

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Thus the special fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$ is

$$\Phi(t) = \Psi(t)\Psi(0)^{-1}$$

$$= \begin{pmatrix} 5e^{7t} & -e^{-4t} \\ e^{7t} & 2e^{-4t} \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 1 & 2 \end{pmatrix}^{-1}$$

$$= \frac{1}{11} \begin{pmatrix} 5e^{7t} & -e^{-4t} \\ e^{7t} & 2e^{-4t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= \frac{1}{11} \begin{pmatrix} 10e^{7t} + e^{-4t} & 5e^{7t} - 5e^{-4t} \\ 2e^{7t} - 2e^{-4t} & e^{7t} + 10e^{-4t} \end{pmatrix}.$$

(b) The solution to the initial value problem is

$$\mathbf{x} = \Phi(t) \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= \frac{1}{11} \begin{pmatrix} 10e^{7t} + e^{-4t} & 5e^{7t} - 5e^{-4t} \\ 2e^{7t} - 2e^{-4t} & e^{7t} + 10e^{-4t} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= \frac{1}{11} \begin{pmatrix} 11e^{-4t} \\ -22e^{-4t} \end{pmatrix}.$$

(c) The phase portrait is

$\square$
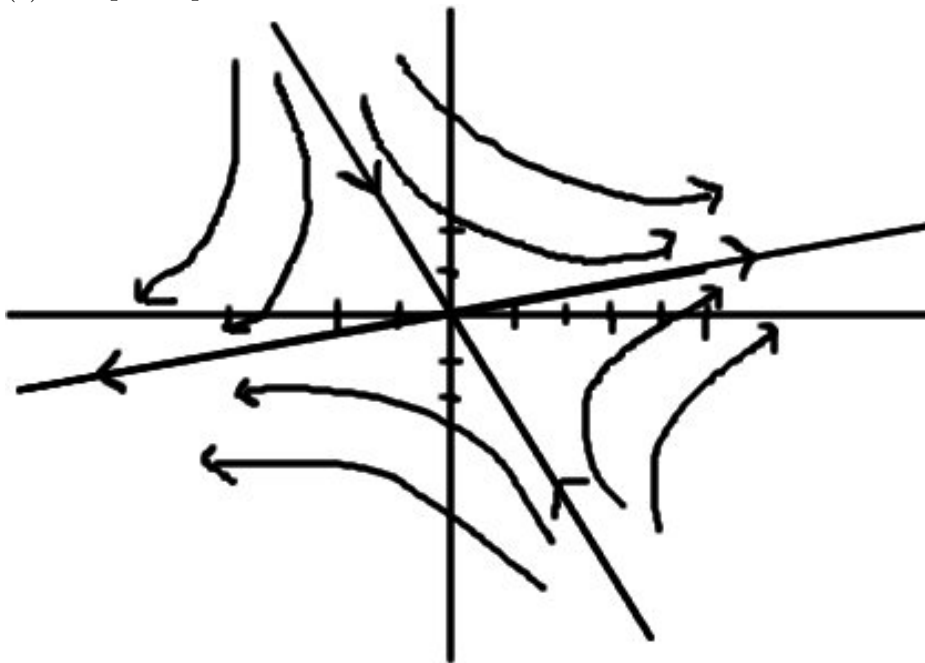
13. (a) Prove that if $A = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$ is diagonal, then $e^{At} = \begin{pmatrix} e^{d_1 t} & & \\ & \ddots & \\ & & e^{d_n t} \end{pmatrix}$.

(b) Prove that if A can be diagonalized as $A = S\Lambda S^{-1}$, then $e^{At} = Se^{\Lambda t}S^{-1}$.

(c) Compute e^{At} using (b) where A is the matrix in problem 11b.

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Solution. (a) We use the definition of e^{At} given by

$$e^{At} = I + \sum_{m=1}^{\infty} \frac{A^m t^m}{m!}.$$

Since A is diagonal, A^m is diagonal with diagonal entries $d_1^m, \ldots, d_n^m$. Plugging this in for A^m above and rewriting the sum, we have

$$e^{At} = I + \sum_{m=1}^{\infty} \begin{pmatrix} \frac{d_1^m t^m}{m!} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \frac{d_n^m t^m}{m!} \end{pmatrix}$$

$$= I + \begin{pmatrix} \sum_{m=1}^{\infty} \frac{d_1^m t^m}{m!} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \sum_{m=1}^{\infty} \frac{d_n^m t^m}{m!} \end{pmatrix}.$$

Finally, we can combine this matrix with the identity, so we get a matrix with diagonal entries as series, which we recognize from calculus as the Taylor series for $e^{d_1 t}, \ldots, e^{d_n t}$ as required.

(b) First, if $A = S\Lambda S^{-1}$, then $A^k = S\Lambda^k S^{-1}$. Thus

$$e^{At} = I + \sum_{k=1}^{\infty} \frac{(At)^k}{k!}$$

$$= I + \sum_{k=1}^{\infty} \frac{S\Lambda^k S^{-1} t^k}{k!}$$

$$= SIS^{-1} + \sum_{k=1}^{\infty} \frac{S(\Lambda t)^k S^{-1}}{k!}$$

$$= S\left(I + \sum_{k=1}^{\infty} \frac{(\Lambda t)^k}{k!}\right) S^{-1}$$

$$= Se^{\Lambda t} S^{-1}$$

as required.

(c) Using the eigenvectors found in 11b, we can diagonalize A as

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Thus by (b) and (a),

$$e^{At} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 2e^{5t} + e^{2t} & 2e^{5t} - 2e^{2t} \\ e^{5t} - e^{2t} & e^{5t} + 2e^{2t} \end{pmatrix}.$$

$\square$

14. (a) Prove that if $\mathbf{u} + i\mathbf{v}$ is a complex solution of the system $\mathbf{x}' = A\mathbf{x}$, then both $\mathbf{u}$ and $\mathbf{v}$ are real solutions.

(b) Solve the following system.

$$\mathbf{x}' = \begin{pmatrix} 5 & 10 \\ -2 & -3 \end{pmatrix} \mathbf{x}$$

Solution. (a) For $\mathbf{u} + i\mathbf{v}$ to be a solution of $\mathbf{x}' = A\mathbf{x}$, we must have

$$(\mathbf{u} + i\mathbf{v})' = A(\mathbf{u} + i\mathbf{v})$$
$$\mathbf{u}' + i\mathbf{v}' = A\mathbf{u} + iA\mathbf{v}.$$

Comparing the real and imaginary parts on both sides, we must have $\mathbf{u}' = A\mathbf{u}$ and $\mathbf{v}' = A\mathbf{v}$, showing that $\mathbf{u}$ and $\mathbf{v}$ themselves are solutions of $\mathbf{x}' = A\mathbf{x}$.

(b) The eigenvalues of A are $1 \pm 2i$, and an eigenvector corresponding to $1 + 2i$ is

$$\begin{pmatrix} 10 \\ -4 + 2i \end{pmatrix}.$$

This gives us a complex solution, which we rewrite as

$$\begin{pmatrix} 10 \\ -4 + 2i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} 10 \\ -4 + 2i \end{pmatrix} e^t (\cos 2t + i \sin 2t)$$

$$= \begin{pmatrix} 10 \cos 2t + i 10 \sin 2t \\ -4 \cos 2t - 2 \sin 2t + 2i \cos 2t - 4i \sin 2t \end{pmatrix} e^t$$

$$= \begin{pmatrix} 10 \cos 2t \\ -4 \cos 2t - 2 \sin 2t \end{pmatrix} e^t + i \begin{pmatrix} 10 \sin 2t \\ -4 \sin 2t + 2 \cos 2t \end{pmatrix} e^t.$$

Hence, using the real and imaginary parts, we find that the general solution of the system is

$$\mathbf{x} = c_1 \begin{pmatrix} 10 \cos 2t \\ -4 \cos 2t - 2 \sin 2t \end{pmatrix} e^t + c_2 \begin{pmatrix} 10 \sin 2t \\ -4 \sin 2t + 2 \cos 2t \end{pmatrix} e^t.$$

□

15. (a) Prove that if $\mathbf{x} = \xi t e^{rt} + \eta e^{rt}$ is a solution of the system $\mathbf{x}' = A\mathbf{x}$, then r is an eigenvalue of A, ξ is an associated eigenvector, and η satisfies $(A - rI)\eta = \xi$.

(b) Solve the following system.

$$\mathbf{x}' = \begin{pmatrix} -3 & -2 \\ 2 & -7 \end{pmatrix} \mathbf{x}$$

Solution. (a) For $\xi t e^{rt} + \eta e^{rt}$ to be a solution of $\mathbf{x}' = A\mathbf{x}$, we must have

$$(\xi t e^{rt} + \eta e^{rt})' = A(\xi t e^{rt} + \eta e^{rt})$$
$$r\xi t e^{rt} + \xi e^{rt} + r\eta e^{rt} = A(\xi t e^{rt}) + A(\eta e^{rt})$$
$$r\xi t e^{rt} + (\xi + r\eta)e^{rt} = t e^{rt} A\xi + e^{rt} A\eta.$$

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Hence comparing the te^{rt} and e^{rt} terms on both sides, we have $A\xi = r\xi$ and $A\eta = r\eta + \xi$. The first equation says that r is an eigenvalue of A and ξ is an associated eigenvector, and the second equation can be rewritten as $(A - rI)\eta = \xi$ as required.

(b) The only eigenvalue of A is -5, and a corresponding eigenvector is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus one solution of the system is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-5t}.$$

Now, solving

$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for η, we find one possibility is

$$\eta = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}.$$

Hence the general solution of the system is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-5t} + c_2 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-5t} + \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} e^{-5t} \right).$$

$\square$

16. Find the eigenvalues and eigenfunctions of the following boundary value problem.

$$y'' + \lambda y = 0, \quad y(L) = 0, \ y'(0) = 0$$

Solution. For we consider the case where $\lambda = \mu^2 > 0$. The general solution to $y'' + \mu^2 y = 0$ is then

$$y = c_1 \cos \mu x + c_2 \sin \mu x.$$

Now, the first boundary condition $y(L) = 0$ implies that

$$c_1 \cos \mu L + c_2 \sin \mu L = 0.$$

We cannot conclude anything about c_1 and c_2 yet so we use the second condition. Since

$$y' = -c_1 \mu \sin \mu x + c_2 \mu \cos \mu x,$$

the condition $y'(0) = 0$ implies that

$$-c_1 \mu \sin 0 + c_2 \mu \cos 0 = 0$$

so $c_2 = 0$. Thus, going back to the first condition, we must have

$$c_1 \cos \mu L = 0.$$

Since we are looking for nontrivial solutions, we must have $\cos \mu L = 0$. For this to be true, μL must be one of $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ — i.e. we must have

$$\mu L = \frac{(2n-1)\pi}{2} \text{ for } n = 1, 2, 3, dotsc.$$

Hence $\mu = \frac{(2n-1)\pi}{2L}$ so the positive eigenvalues are

$$\lambda_n = \frac{(2n-1)^2 \pi^2}{4L^2} \text{ for } n = 1, 2, dotsc$$

and eigenfunction for these are

$$y_n = \cos \frac{(2n-1)\pi x}{2L} \text{ for } n = 1, 2, dotsc.$$

Second we consider the case where $\lambda = 0$. Then the general solution to $y'' = 0$ is $y = c_1 x + c_2$. The first condition $y(L) = 0$ implies that $c_1 L + c_2 = 0$ and since $y' = c_1$, the second condition $y'(0) = 0$ implies that $c_1 = 0$. Hence $c_2 = 0$ from the first condition so there are no nontrivial solutions in this case — i.e. 0 is not an eigenvalue.

Finally we consider the case where $\lambda = -\mu^2 < 0$. The general solution to $y'' - \mu^2 y = 0$ is then

$$y = c_1 \cosh \mu x + c_2 \sinh \mu x.$$

Since

$$y' = c_1 \mu \sinh \mu x + c_2 \mu \cosh \mu x,$$

the boundary condition $y'(0) = 0$ implies that

$$c_1 \mu \sinh 0 + c_2 \mu \cosh 0 = 0$$

so we must have $c_2 = 0$. Then, the condition $y(L) = 0$ implies that

$$c_1 \cosh \mu L = 0$$

and since $\cosh x$ is never 0, it follows that $c_1 = 0$. Thus there are no nontrivial solutions in this case either so the equation has no negative eigenvalues. $\square$

17. (a) Derive the formulas for the Fourier coefficients of a function with period $2L$.

(b) Find the Fourier series of the following function and draw the graph of the function to which the Fourier series converges for three periods.

$$f(x) = \begin{cases} 0, & -2 \le x < 0 \\ 1 - x, & 0 \le x < 2 \end{cases}$$

Solution. (a) Let f be a function of period $2L$. Throughout, suppose that

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right) \tag{1}$$

is the Fourier series of f. Also, let $\langle , \rangle$ be the inner product

$$\langle f, g \rangle = \int_{-L}^{L} f(x)g(x)\,dx$$

and recall the orthogonality relations

$$\left\langle \cos\frac{m\pi x}{L}, \cos\frac{n\pi x}{L} \right\rangle = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \neq 0 \\ 2L & \text{if } m = n = 0 \end{cases}$$

$$\left\langle \cos\frac{m\pi x}{L}, \sin\frac{n\pi x}{L} \right\rangle = 0 \text{ for any } m \text{ and } n$$

$$\left\langle \sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L} \right\rangle = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}.$$

First, we derive the formula for a_0. Take the inner product of both sides of (1) with $\cos 0 = 1$ and distribute to get

$$\langle f(x), \cos 0 \rangle = \left\langle \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L} \right), \cos 0 \right\rangle$$

$$= \frac{a_0}{2} \langle \cos 0, \cos 0 \rangle + \sum_{m=1}^{\infty} \left(a_m \left\langle \cos\frac{m\pi x}{L}, \cos 0 \right\rangle + b_m \left\langle \sin\frac{m\pi x}{L}, \cos 0 \right\rangle \right).$$

By the orthogonality relations, the only inner product on the right side which is nonzero is $\langle \cos 0, \cos 0 \rangle$. Hence the only nonzero term on the right side is $\frac{a_0}{2} \langle \cos 0, \cos 0 \rangle$ so

$$\langle f(x), \cos 0 \rangle = \frac{a_0}{2} \langle \cos 0, \cos 0 \rangle$$

and thus

$$a_0 = 2\frac{\langle f(x), \cos 0 \rangle}{\langle \cos 0, \cos 0 \rangle}.$$

Using the orthogonality relations and the definition of the inner product, this becomes

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx.$$

Second, we derive the formula for $a_n, n \neq 0$. Take the inner product of both sides of (1) with $\cos\frac{n\pi x}{L}$ and distribute to get

$$\left\langle f(x), \cos\frac{n\pi x}{L} \right\rangle = \left\langle \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L} \right), \cos\frac{n\pi x}{L} \right\rangle$$

$$= \frac{a_0}{2} \left\langle \cos 0, \cos\frac{n\pi x}{L} \right\rangle + \sum_{m=1}^{\infty} \left(a_m \left\langle \cos\frac{m\pi x}{L}, \cos\frac{n\pi x}{L} \right\rangle + b_m \left\langle \sin\frac{m\pi x}{L}, \cos\frac{n\pi x}{L} \right\rangle \right).$$

By the orthogonality relations, the only inner product on the right side which is nonzero is $\left\langle \cos \frac{n\pi x}{L}, \cos \frac{n\pi x}{L} \right\rangle$, so the only nonzero term on the right side is $a_n \left\langle \cos \frac{n\pi x}{L}, \cos \frac{n\pi x}{L} \right\rangle$ so

$$\left\langle f(x), \cos \frac{n\pi x}{L} \right\rangle = a_n \left\langle \cos \frac{n\pi x}{L}, \cos \frac{n\pi x}{L} \right\rangle$$

and thus

$$a_n = \frac{\left\langle f(x), \cos \frac{n\pi x}{L} \right\rangle}{\left\langle \cos \frac{n\pi x}{L}, \cos \frac{n\pi x}{L} \right\rangle}.$$

Using the orthogonality relations and the definition of the inner product, this becomes

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx.$$

Finally, we derive the formula for b_n. Take the inner product of both sides of (1) with $\sin \frac{n\pi x}{L}$ and distribute to get

$$\left\langle f(x), \sin \frac{n\pi x}{L} \right\rangle = \left\langle \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right), \sin \frac{n\pi x}{L} \right\rangle$$

$$= \frac{a_0}{2} \left\langle \cos 0, \sin \frac{n\pi x}{L} \right\rangle + \sum_{m=1}^{\infty} \left(a_m \left\langle \cos \frac{m\pi x}{L}, \sin \frac{n\pi x}{L} \right\rangle + b_m \left\langle \sin \frac{m\pi x}{L}, \sin \frac{n\pi x}{L} \right\rangle \right).$$

By the orthogonality relations, the only inner product on the right side which is nonzero is $\left\langle \sin \frac{n\pi x}{L}, \sin \frac{n\pi x}{L} \right\rangle$, so the only nonzero term on the right side is $b_n \left\langle \sin \frac{n\pi x}{L}, \sin \frac{n\pi x}{L} \right\rangle$ so

$$\left\langle f(x), \sin \frac{n\pi x}{L} \right\rangle = b_n \left\langle \sin \frac{n\pi x}{L}, \sin \frac{n\pi x}{L} \right\rangle$$

and thus

$$b_n = \frac{\left\langle f(x), \sin \frac{n\pi x}{L} \right\rangle}{\left\langle \sin \frac{n\pi x}{L}, \sin \frac{n\pi x}{L} \right\rangle}.$$

Using the orthogonality relations and the definition of the inner product, this becomes

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx.$$

Note: This all may seem like a lot of work and difficult to remember, but it is (relatively) simple if you notice the similarity between this and problem 7 — the point is that the functions $\cos \frac{m\pi x}{L}$ for $m = 0, 1, 2, \ldots$ and $\sin \frac{n\pi x}{L}$ for $n = 1, 2, 3, \ldots$ are *orthogonal* under this inner product, so the formulas for the Fourier coefficients give us the *coordinates* of a function relative to this basis — when we are finding a Fourier series, we are expressing a function as a linear combination of the orthogonal vectors $\cos \frac{m\pi x}{L}$ for $m = 0, 1, 2, \ldots$ and $\sin \frac{n\pi x}{L}$ for $n = 1, 2, 3, \ldots$
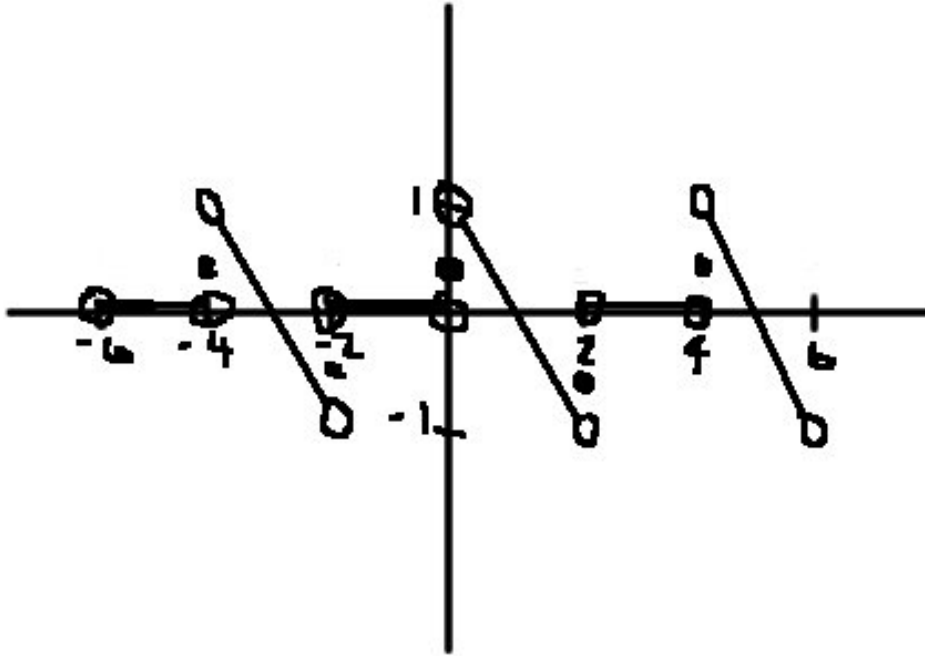
(b) We compute the Fourier coefficients as follows:

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\, dx$$

$$= \frac{1}{2} \int_{0}^{2} (1-x)\, dx$$

$$= \frac{1}{2} \left(x - \frac{x^2}{2} \right) \Big|_{0}^{2}$$

$$= 0$$

$$a_m = \frac{1}{2} \int_{0}^{2} (1-x) \cos \frac{m\pi x}{2}\, dx$$

$$= \frac{1}{m\pi} (1-x) \sin \frac{m\pi x}{2} \Big|_{0}^{2} + \frac{1}{m\pi} \int_{0}^{2} \sin \frac{m\pi x}{2}\, dx$$

$$= -\frac{2}{(m\pi)^2} \cos \frac{m\pi x}{2} \Big|_{0}^{2}$$

$$= \frac{2}{(m\pi)^2} (1 - \cos m\pi)$$

$$b_m = \frac{1}{2} \int_{0}^{2} (1-x) \sin \frac{m\pi x}{2}\, dx$$

$$= -\frac{1}{m\pi} (1-x) \cos \frac{m\pi x}{2} \Big|_{0}^{2} - \frac{1}{m\pi} \int_{0}^{2} \cos \frac{m\pi x}{2}\, dx$$

$$= \frac{1}{m\pi} (\cos m\pi + 1) - \frac{2}{(m\pi)^2} \sin \frac{m\pi x}{2} \Big|_{0}^{2}$$

$$= \frac{1}{m\pi} (\cos m\pi + 1).$$

Thus the Fourier series of f is

$$\sum_{m=1}^{\infty} \left(\frac{2}{(m\pi)^2} (1 - \cos m\pi) \cos \frac{m\pi x}{2} + \frac{1}{m\pi} (\cos m\pi + 1) \sin \frac{m\pi x}{2} \right)$$

and the graph of the function it converges to is

18. Let $f(x) = 1 + x$ for $0 < x < 1$.
    (a) Find the Fourier sine series of the given function.
    (b) Find the Fourier cosine series of the given function.

Solution. (a) We are using the odd extension of f, so we only need to compute the b_m coefficients. We have

$$b_m = \frac{2}{L} \int_0^L (1+x) \sin \frac{m\pi x}{L} \, dx$$
$$= 2 \int_0^1 (1+x) \sin m\pi x \, dx$$
$$= -\frac{2}{m\pi} (1+x) \cos m\pi x \Big|_0^1 + \frac{2}{m\pi} \int_0^1 \cos m\pi x \, dx$$
$$= \frac{2}{m\pi} (1 - 2 \cos m\pi).$$

Hence the Fourier sine series of f is

$$\sum_{m=1}^{\infty} \frac{2}{m\pi} (1 - 2 \cos m\pi) \sin m\pi x.$$

(b) We are using the even extension of f, so we only need to compute a_0 and the

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a_m coefficients. We have

$$a_0 = \frac{2}{L} \int_0^L (1+x)\, dx$$

$$= 2 \int_0^1 (1+x)\, dx$$

$$= 2 \left(x + \frac{x^2}{2} \right) \Big|_0^1$$

$$= 3$$

$$a_m = \frac{2}{L} \int_0^L (1+x) \cos \frac{m\pi x}{L}\, dx$$

$$= 2 \int_0^1 (1+x) \cos m\pi x\, dx$$

$$= \frac{2}{m\pi} (1+x) \sin m\pi x \Big|_0^1 - \frac{2}{m\pi} \int_0^1 \sin m\pi x\, dx$$

$$= \frac{2}{(m\pi)^2} \cos m\pi x \Big|_0^1$$

$$= \frac{2}{(m\pi)^2} (\cos m\pi - 1).$$

Hence the Fourier cosine series of f is

$$\frac{3}{2} + \sum_{m=1}^{\infty} \frac{2}{(m\pi)^2} (\cos m\pi - 1) \cos m\pi x.$$

$\square$

19. Solve the following heat equation (from scratch).

$$4u_{xx} = u_t, \quad 0 < x < 2$$

with boundary conditions
$$u(0,t) = 0 = u(2,t)$$

and initial condition $u(x,0) = 2 \sin \pi x - 5 \sin 4\pi x$.

Solution. First we suppose that our function u can be written as

$$u(x,t) = X(x)T(t)$$

for some functions X and T. Plugging this into the heat equation gives

$$4X''T = XT'.$$

Now, we separate variables to get

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{4T(t)}.$$

Holding t fixed on the right side, we see that the left side then has to be a constant for all x and thus the right side is the same constant for all t. Call this constant $-\lambda$, so

$$\frac{X''}{X} = -\lambda \text{ and } \frac{T'}{4T} = -\lambda.$$

Rewriting these equations we get the ordinary differential equations

$$X'' + \lambda X = 0 \text{ and } T' + 4\lambda T = 0.$$

Now, the boundary conditions $u(0,t) = 0$ and $u(2,t) = 0$ give

$$X(0)T(t) = 0 \text{ and } X(2)T(t) = 0$$

respectively. We assume that T is not 0 since we are looking for nontrivial solutions, so we conclude that $X(0) = 0$ and $X(2) = 0$. Thus we have the following boundary value problem for X:

$$X'' + \lambda X = 0, \ X(0) = 0, \ X(2) = 0.$$

Going through the process of finding the eigenvalues and eigenfunctions, we find that the only eigenvalues are $\lambda_n = \frac{n^2\pi^2}{4}$ for $n = 1, 2, \ldots$ and corresponding eigenfunctions are $X_n = \sin \frac{n\pi x}{2}$. For each of these eigenvalues, we have the following first order differential equation for T:

$$T' + 4\lambda_n T = 0$$

which has the nontrivial solution

$$T_n = e^{-4\lambda_n t} = e^{-n^2\pi^2 t}.$$

Thus, for each $n = 1, 2, \ldots$, we have the following nontrivial solution of the heat equation:

$$u_n(x,t) = X_n(x)T_n(t) = e^{-n^2\pi^2 t} \sin \frac{n\pi x}{2}.$$

The general solution is then an arbitrary linear combination of these:

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin \frac{n\pi x}{2}.$$

Finally, the initial condition

$$u(x,0) = 2\sin \pi x - 5\sin 4\pi x$$

implies that

$$2\sin \pi x - 5\sin 4\pi x = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin \frac{n\pi x}{2}.$$

Thus the b_n must be the Fourier sine coefficients of $f(x) = 2\sin \pi x - 5\sin 4\pi x$. We can compute these using the integral formulas as usual, but since $f(x)$ is already in the form

of a Fourier sine series, there is an easier way to compute these coefficients. First we rewrite $f(x)$ so that each sin term has $L = 2$ in the denominator:

$$f(x) = 2\sin\frac{2\pi x}{2} - 5\sin\frac{8\pi x}{2}.$$

From this we can see that $b_2 = 2$, $b_8 = -5$, and every other b_n is 0. Plugging these into the formula for $u(x, t)$, we see that the solution to our heat equation is

$$u(x, t) = 2e^{-4\pi^2 t}\sin\frac{2\pi x}{2} - 5e^{-64\pi^2 t}\sin\frac{8\pi x}{2}.$$

$\square$

20. Solve the following partial differential equation.

$$tu_{xx} = u_t, \quad 0 < x < \pi$$

with boundary conditions
$$u(0, t) = 0 = u(\pi, t)$$

and initial condition $u(x, 0) = x$.

Solution. As in problem 19, we assume that $u(x, t) = X(x)T(t)$. Plugging this into the given partial differential equation gives

$$tX''T = XT'$$

and separating variables we have
$$\frac{X''}{X} = \frac{T'}{tT}.$$

As in problem 19, we can then conclude that both sides must be constant, say $-\lambda$. Setting both sides equal to this constant and rewriting the equations we get the ordinary differential equations
$$X'' + \lambda X = 0 \text{ and } T' + \lambda tT = 0.$$

From the boundary conditions $u(0, t) = 0 = u(\pi, t)$ we get the following boundary value problem for X:

$$X'' + \lambda X = 0, \ X(0) = 0, \ X(\pi) = 0.$$

The only eigenvalues of this are $\lambda_n = n^2$ for $n = 1, 2, \ldots$ and eigenfunctions for each of these are $X_n = \sin nx$. Now, for each of these eigenvalues, we have the following first order differential equation for T:

$$T' + \lambda_n tT = 0.$$

We can solve this using separation of variables from Math 1B, and one nontrivial solution is then
$$T_n = e^{-\frac{\lambda_n t^2}{2}} = e^{-\frac{n^2 t^2}{2}}.$$

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Thus for each $n = 1, 2, \ldots$, we have the following solution to the given partial differential equation:

$$u_n(x, t) = X_n T_n = e^{-\frac{n^2 t^2}{2}} \sin nx.$$

The general solution is then an arbitrary linear combination of these:

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2 t^2}{2}} \sin nx.$$

The initial condition $u(x, 0) = x$ implies

$$x = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2 t^2}{2}} \sin nx$$

so the b_n are the Fourier sine coefficients of $f(x) = x$. We compute these as follows (here $L = \pi$):

$$\begin{aligned}
b_n &= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx \\
&= -\frac{2}{n\pi} x \cos nx \Big|_0^{\pi} + \frac{2}{n\pi} \int_0^{\pi} \cos nx \, dx \\
&= -\frac{2}{n\pi} \pi \cos n\pi + \frac{2}{n^2 \pi} \sin nx \Big|_0^{\pi} \\
&= -\frac{2}{n} \cos n\pi.
\end{aligned}$$

Thus the solution to the given partial differential equation with given boundary conditions and initial condition is

$$u(x, t) = \sum_{n=1}^{\infty} -\frac{2 \cos n\pi}{n} e^{-\frac{n^2 t^2}{2}} \sin nx.$$

$\square$