Midterm 2 – Review – Problems

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1 Vector space-stuff

Problem 1

Consider

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find $Rank(A)$, $\dim(Col(A))$, $\dim(Row(A))$, $\dim(Nul(A))$

(b) Find a basis for $Row(A)$ and a basis for $Col(A)$

Problem 2

If $\mathcal{B} = \{-1 + 8t, 1 - 5t\}$ and $\mathcal{C} = \{1 + 4t, 1 + t\}$, find $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$ and use this to calculate $[p]_{\mathcal{B}}$ given $p = 1(1 + 4t) + 4(1 + t)$.

2 Diagonalization

Problem 3

Find a nonzero vector $\mathbf{v}$ such that $\lim_{n\to\infty} A^n\mathbf{v} = \mathbf{0}$, where:

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

3 Linear transformations

Problem 4

Let $T : P_2 \longrightarrow M_{2\times 2}$ be the following linear transformation:

$$T(p) = \begin{bmatrix} p(0) & p'(1) \\ p''(2) & p(0) \end{bmatrix}$$

Find the matrix T with respect to the basis $\mathcal{B} = \{1, t, t^2\}$ of P_2 and the basis $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ of $M_{2\times 2}$

4 Orthogonal projections and Least-squares

Problem 5

Find the least squares solution and the least-squares error to $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Do it directly, and do it using orthogonal projections.

5 Differential Equations

Problem 6

Find the largest interval (a, b) on which the following differential equation has a unique solution:

$$\tan(t)y'' + (t-1)y' + 3y = \tan^2(t)$$

with $y\left(\frac{1}{2}\right) = 0$, $y'\left(\frac{1}{2}\right) = 1$.

Problem 7

Find all the solutions to $y'''' - 5y''' - 2y'' + 24y' = 0$ such that $\lim_{t\to\infty} y(t) = 54$

Problem 8

Guess the form of a particular solution of:

$$D^3(D-1)^2\left(D^2 - 2D + 5\right)(y) = f(t)$$

where:

(a) $f(t) = t^2 e^t$

(b) $f(t) = te^t \cos(2t)$

(c) $f(t) = e^t \cos(3t)$

(d) $f(t) = e^{2t} \cos(2t)$

(e) $f(t) = t^2 \sin(2t)$

(f) $f(t) = t$