MATH 54 – MOCK FINAL EXAM – SOLUTIONS

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1. *(10 points, 2 points each)*

Label the following statements as **T** or **F**. **Write your answers in the box below!**

NOTE: In this question, you do **NOT** have to show your work! Don't spend *too* much time on each question!

(a) **FALSE** If Q has orthogonal columns, then Q is an orthogonal matrix

(The columns of Q have to be ortho*normal*)

(b) **TRUE** If $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{x}$ on W, then $\mathbf{x}-\hat{\mathbf{x}}$ is always orthogonal to $\hat{\mathbf{x}}$.

(Draw a picture!)

(c) **FALSE** The least-squares solution $\widetilde{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ has the property that $\|A\mathbf{x} - \mathbf{b}\| \leq \|A\widetilde{\mathbf{x}} - \mathbf{b}\|$ for every $\mathbf{x}$

(It has the property that $\|A\widetilde{\mathbf{x}} - \mathbf{b}\| \leq \|A\mathbf{x} - \mathbf{b}\|$, i.e. it *minimizes* the least-squares error)

(d) **FALSE** If a set $\mathcal{B}$ is orthogonal, then $\mathcal{B}$ is linearly independent

(It *could* contain the **0**-vector! However, if you ignore the **0**-vector, then it is linearly independent)

(e) **FALSE** $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = ac$ defines a dot/inner product on $\mathbb{R}^2$.

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$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (0)(0) = 0 \quad \text{even though} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \mathbf{0}$$

(The point is that the last property of dot products is usually very good to check if something is not a dot product!)

2. *(15 points)* Use the Gram-Schmidt process to find an orthonormal basis for W, where:

$$W = Span \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

Define:

$$\mathbf{u_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

First, let's find an *orthogonal* basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ for W:

<u>Step 1</u>: Let $\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

<u>Step 2</u>: Calculate:

$$\hat{\mathbf{u_2}} = \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1} = \frac{2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

And let:

$$\mathbf{v_2} = \mathbf{u_2} - \hat{\mathbf{u_2}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Note: You can easily check that $\mathbf{v_1} \cdot \mathbf{v_2} = 0$. This is a good way to check if you got the right answer!

Step 3: Calculate:

$$\hat{\mathbf{u_3}} = \left(\frac{\mathbf{u_3} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1} + \left(\frac{\mathbf{u_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}} \right) \mathbf{v_2} = \frac{2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

And let:

$$\mathbf{v_3} = \mathbf{u_3} - \hat{\mathbf{u_3}} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Note: You can check that $\mathbf{v_1} \cdot \mathbf{v_2} = 0$ and $\mathbf{v_1} \cdot \mathbf{v_3} = 0$.

Step 4: Normalize:

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{w_3} = \frac{\mathbf{v_3}}{\|\mathbf{v_3}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Answer:

$$\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\} = \left\{ \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}$$

3. *(15 points)* Find the least-squares solution and least-squares error to the following (inconsistent) system of equations $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

We need to solve:

$$A^T A \widetilde{\mathbf{x}} = A^T \mathbf{b}$$

But:

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

And

$$A^T \mathbf{b} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

Hence we need to solve:

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

Now you can either compute the inverse of the matrix, or row-reduce:

$$\begin{aligned} \begin{bmatrix} 17 & 1 & 19 \\ 1 & 5 & 11 \end{bmatrix} &\to \begin{bmatrix} 17 & 1 & 19 \\ 0 & -84 & -168 \end{bmatrix} \\ &\to \begin{bmatrix} 17 & 1 & 19 \\ 0 & 1 & 2 \end{bmatrix} \\ &\to \begin{bmatrix} 17 & 0 & 17 \\ 0 & 1 & 2 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{aligned}$$

(For the first row-reduction, I substracted 17 times the second row from the first! Also, I apologize for the messy algebra, the algebra on the final will be simpler)

Hence, we get:

$$\widetilde{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

<u>Least-squares error:</u>

$$\begin{aligned} \|A\widetilde{\mathbf{x}} - \mathbf{b}\| &= \left\| \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} 2 \\ 4 \\ -8 \end{bmatrix} \right\| \\ &= \sqrt{2^2 + 4^2 + (-8)^2} \\ &= \sqrt{4 + 16 + 64} \\ &= \sqrt{84} \end{aligned}$$

4. *(30 points)* Solve the following heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = x & 0 < x < 1 \end{cases}$$

Note: You may not use **ANY** for the formulas given in the book! You have to do it from scratch, including the 3 cases.

Note: The following formula might be useful:

$$\int_{-1}^{1} \cos^2(\pi m x) = \int_{-1}^{1} \sin^2(\pi m x) = 1$$

Step 1: Separation of variables. Suppose:

$$u(x,t) = X(x)T(t) \tag{1}$$

Plug (1) into the differential equation (), and you get:

$$\begin{aligned}(X(x)T(t))_t =& (X(x)T(t))_{xx}\\ X(x)T'(t) =& X''(x)T(t)\end{aligned}$$

Rearrange and get:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} \tag{2}$$

Now $\frac{X''(x)}{X(x)}$ *only* depends on x, but by (2) *only* depends on t, hence it is constant:

$$\begin{aligned}\frac{X''(x)}{X(x)} =& \lambda\\ X''(x) =& \lambda X(x)\end{aligned} \tag{3}$$

Also, we get:

$$\begin{aligned}\frac{T'(t)}{T(t)} =& \lambda\\ T'(t) =& \lambda T(t)\end{aligned} \tag{4}$$

but we'll only deal with that later (Step 4)

Step 2: Consider (3):

$$X''(x) = \lambda X(x)$$

Note: Always start with $X(x)$, do **NOT** touch $T(t)$ until right at the end!

Now use the **boundary conditions** in ():

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u(1,t) = X(1)T(t) = 0 \Rightarrow X(1)T(t) = 0 \Rightarrow X(1) = 0$$

Hence we get:

$$\begin{cases} X''(x) = \lambda X(x) \\ X(0) = 0 \\ X(1) = 0 \end{cases} \tag{5}$$

Step 3: Eigenvalues/Eigenfunctions. The auxiliary polynomial of (5) is $p(\lambda) = r^2 - \lambda$

Now we need to consider 3 cases:

<u>Case 1:</u> $\lambda > 0$, then $\lambda = \omega^2$, where $\omega > 0$

Then:

$$r^2 - \lambda = 0 \Rightarrow r^2 - \omega^2 = 0 \Rightarrow r = \pm\omega$$

Therefore:

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$

Now use $X(0) = 0$ and $X(1) = 0$:

$$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A \Rightarrow X(x) = Ae^{\omega x} - Ae^{-\omega x}$$

$$X(1) = 0 \Rightarrow Ae^{\omega} - Ae^{-\omega} = 0 \Rightarrow Ae^{\omega} = Ae^{-\omega} \Rightarrow e^{\omega} = e^{-\omega} \Rightarrow \omega = -\omega \Rightarrow \omega = 0$$

But this is a **contradiction**, as we want $\omega > 0$.

<u>Case 2:</u> $\lambda = 0$, then $r = 0$, and:

$$X(x) = Ae^{0x} + Bxe^{0x} = A + Bx$$

And:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = Bx$$

$$X(1) = 0 \Rightarrow B = 0 \Rightarrow X(x) = 0$$

Again, a **contradiction** (we want $X \not\equiv 0$, because otherwise $u(x,t) \equiv 0$)

<u>Case 3:</u> $\lambda < 0$, then $\lambda = -\omega^2$, and:

$$r^2 - \lambda = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm \omega i$$

Which gives:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

Again, using $X(0) = 0$, $X(1) = 0$, we get:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B\sin(\omega x)$$

$$X(1) = 0 \Rightarrow B\sin(\omega) = 0 \Rightarrow \sin(\omega) = 0 \Rightarrow \omega = \pi m, \quad (m = 1, 2, \cdots)$$

This tells us that:

$$\begin{aligned} &\text{Eigenvalues:} \lambda = -\omega^2 = -(\pi m)^2 \quad (m = 1, 2, \cdots) \\ &\text{Eigenfunctions:} X(x) = \sin(\omega x) = \sin(\pi m x) \end{aligned} \tag{6}$$

Step 4: Deal with (4), and remember that $\lambda = -(\pi m)^2$:

$$T'(t) = \lambda T(t) \Rightarrow T(t) = Ae^{\lambda t} = T(t) = \widetilde{A_m} e^{-(\pi m)^2 t} \qquad m = 1, 2, \cdots$$

Note: Here we use $\widetilde{A_m}$ to emphasize that $\widetilde{A_m}$ depends on m.

Step 5: Take linear combinations:

$$u(x,t) = \sum_{m=1}^{\infty} T(t)X(x) = \sum_{m=1}^{\infty} \widetilde{A_m} e^{-(\pi m)^2 t} \sin(\pi m x) \tag{7}$$

Step 6: Use the initial condition $u(x,0) = x$ in ():

$$u(x,0) = \sum_{m=1}^{\infty} \widetilde{A_m} \sin(\pi m x) = x \qquad \text{on}(0,1) \tag{8}$$

Now we want to express x as a linear combination of sines, so we have to use a **sine series** (that's why we used $\widetilde{A_m}$ instead of A_m):

$$\begin{aligned}
\widetilde{A_m} &= \frac{2}{1}\int_0^1 x\sin(\pi m x)dx \\
&= 2\left(\left[-x\frac{\cos(\pi m x)}{\pi m}\right]_0^1 - \int_0^1 -\frac{\cos(\pi m x)}{\pi m}dx\right) \\
&= 2\left(-\frac{\cos(\pi m)}{\pi m} + \int_0^1 \frac{\cos(\pi m x)}{\pi m}dx\right) \\
&= 2\left(-\frac{(-1)^m}{\pi m} + \left[\frac{\sin(\pi m x)}{(\pi m)^2}\right]_0^1\right) \\
&= \frac{2(-1)^{m+1}}{\pi m} \qquad (m = 1, 2, \cdots)
\end{aligned}$$

Step 7: Conclude using (9)

(9) $$u(x,t) = \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{\pi m} e^{-(\pi m)^2 t}\sin(\pi m x)$$

5. *(15 points)*

(a) *(10 points)* Find the Fourier cosine series of $f(x) = x^2$ on $(0, \pi)$

We want to find A_m such that:

$$x^2 \text{``} = \text{''} \sum_{m=0}^{\infty} A_m \cos(mx)$$

Now 'evenify' f to get $\widetilde{f}$ (see lecture), and then:

$$A_0 = \frac{\int_{-\pi}^{\pi} \widetilde{f}(x)}{\int_{-\pi}^{\pi} 1^2} = \frac{2}{2\pi}\int_0^{\pi} x^2 = \frac{1}{\pi}\left(\frac{\pi^3}{3}\right) = \frac{\pi^2}{3}$$

And:

$$A_m = \frac{\int_{-\pi}^{\pi} \widetilde{f}(x)\cos(mx)}{\int_{-\pi}^{\pi} \cos^2(mx)} = \frac{2}{\pi}\int_0^{\pi} x^2\cos(mx)$$

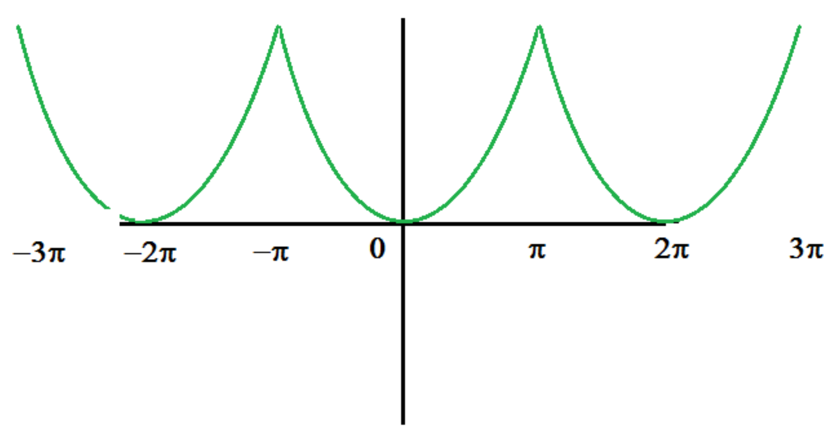
To evaluate this, use tabular integration (see lecture), and you get:

$$\begin{aligned}A_m =&\frac{2}{\pi}\int_0^{\pi} x^2\cos(mx)dx\\ =&\frac{2}{\pi}\left[+x^2\left(\frac{\sin(mx)}{m}\right)-2x\left(\frac{-\cos(mx)}{m^2}\right)+2\left(\frac{-\sin(mx)}{m^3}\right)\right]_0^{\pi}\\ =&\frac{2}{\pi}\left(2\pi\frac{\cos(\pi m)}{m^2}\right)\\ =&\frac{4(-1)^m}{m^2}\end{aligned}$$

(b) *(5 points)* Draw the graph of the function to which the above Fourier series $\mathcal{F}$ converges to on $(-3\pi, 3\pi)$

Notice that since x^2 is even on $(-\pi,\pi)$, $\widetilde{f}(x) = f(x) = x^2$, then, since there are no jumps and the values at the endpoints are the same, we get that $\mathcal{F}(x) = f(x) = x^2$ on $(-\pi,\pi)$ and to get the graph of $\mathcal{F}$ over $(-3\pi, 3\pi)$, just 'repeat' the graph of x^2 one more time on the right, and one more time on the left!

As a result, you get the following picture:
54/Math 54 Summer/Exams/Mockfinalgraph.png



6. *(15 points)*

Prove the parallelogram identity:

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

Note: Do it in general, not just for $\mathbb{R}^n$

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 =& (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\
=& \mathbf{u}\cdot\mathbf{u} + \cancel{\mathbf{u}\cdot\mathbf{v}} + \cancel{\mathbf{v}\cdot\mathbf{u}} + \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} - \cancel{\mathbf{u}\cdot\mathbf{v}} - \cancel{\mathbf{v}\cdot\mathbf{u}} + \mathbf{v}\cdot\mathbf{v} \\
=& 2(\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{v}\cdot\mathbf{v}) \\
=& 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2
\end{aligned}$$