# MATH 54 – FINAL EXAM – SOLUTIONS

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1. *(10 points, 2 points each)*

Label the following statements as **T** or **F**. **Write your answers in the box below!**

**NOTE:** In this question, you do **NOT** have to show your work! Don't spend *too* much time on each question!

(a) **FALSE** If $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{x}$ on $W$, then $\hat{\mathbf{x}}$ is orthogonal to $\mathbf{x}$.
(Draw a picture)

(b) **FALSE** If $\hat{\mathbf{u}}$ is the orthogonal projection of $\mathbf{u}$ on $Span\{\mathbf{v}\}$, then:

$$\hat{\mathbf{u}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{u}$$

(It's $\hat{\mathbf{u}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$, it has to be a multiple of $\mathbf{v}$)

(c) **TRUE** For any (continuous) $f$ and $g$,

$$\left(\int_0^1 f(t)g(t)dt\right)^2 \leq \left(\int_0^1 (f(t))^2 dt\right)\left(\int_0^1 (g(t))^2 dt\right)$$

(This is just the Cauchy-Schwarz inequality with $f\cdot g = \int_0^1 f(t)g(t)dt$:

$$\left|\int_0^1 f(t)g(t)\right| \leq \sqrt{\int_0^1 (f(t))^2 dt}\sqrt{\int_0^1 (g(t))^2 dt}$$

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Now square both sides)

(d) **FALSE** If $\hat{\mathbf{x}}$ is the least-squares solution of $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, then $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{x}$ on $Col(A)$.

(We're not projecting $\mathbf{x}$ onto anything! To find the least-squares solution, project $\mathbf{b}$ onto $Col(A)$ to get $\hat{\mathbf{b}}$ and then find $\hat{\mathbf{x}}$ such that $A\hat{\mathbf{x}} = \mathbf{b}$)

(e) **FALSE** If $Q$ is an orthogonal matrix, then $Q$ is invertible. ($Q$ might not be square!)

2. *(10 points)* Apply the Gram-Schmidt process to find an *orthonormal* basis of $W$, where:

$$W = Span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\} = Span\left\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\right\}$$

Step 1: Let $\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

Step 2: Calculate:

$$\hat{\mathbf{u_2}} = \left(\frac{\mathbf{u_2}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} = \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix}$$

And let:

$$\mathbf{v_2} = \mathbf{u_2} - \hat{\mathbf{u_2}} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{bmatrix} \sim \begin{bmatrix}1\\1\\-2\end{bmatrix}$$

Step 3: Calculate:

$$\hat{\mathbf{u_3}} = \left(\frac{\mathbf{u_3}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} + \left(\frac{\mathbf{u_3}\cdot\mathbf{v_2}}{\mathbf{v_2}\cdot\mathbf{v_2}}\right)\mathbf{v_2} = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

And let:

$$\mathbf{v_3} = \mathbf{u_3} - \hat{\mathbf{u_3}} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\0\end{bmatrix} \sim \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

Step 4: Normalize:

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{w_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix} \quad \mathbf{w_3} = \frac{\mathbf{v_3}}{\|\mathbf{v_3}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

**Answer:**

$$\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}$$

3. *(10 points)* Consider the space $C[-\frac{\pi}{2}, \frac{\pi}{2}]$ with the dot product:

$$f \cdot g = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)g(t)dt$$

Find the orthogonal projection of $f(t) = \cos(x)$ on

$$W = Span\,\{1, \sin(x), \sin(2x)\}$$

And use this to find a function $g$ which is orthogonal to $f$.

$$\begin{aligned}
\hat{f} &= \left(\frac{\cos(t)\cdot 1}{1\cdot 1}\right)1 + \left(\frac{\cos(t)\cdot\sin(t)}{\sin(t)\cdot\sin(t)}\right)\sin(t) + \left(\frac{\cos(t)\cdot\sin(2t)}{\sin(2t)\cdot\sin(2t)}\right)\sin(2t) \\
&= \left(\frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(t)}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}1}\right)1 + \left(\frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(t)\sin(t)}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2(t)}\right)\sin(t) + \left(\frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(t)\sin(2t)}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2(2t)}\right)\sin(2t) \\
&= \left(\frac{[\sin(t)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}{\pi}\right) + \frac{0}{\cdots}\sin(t) + \frac{0}{\cdots}\cos(t) \\
&= \frac{2}{\pi}
\end{aligned}$$

Here we used the facts that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(t)\sin(t) = 0$ and $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(t)\sin(2t) = 0$ which follow from the fact that the integral of an **odd** function over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is 0.

And finally:

$$g(t) = f(t) - \hat{f}(t) = \cos(t) - \frac{\pi}{2}$$

4. *(10 points)* Consider the (inconsistent) system of equations $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

(a) *(5 points)* Find the orthogonal projection of $\mathbf{b}$ on $Col(A)$

Let $\mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{a_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ be the columns of $A$. Then:

$$\begin{aligned} \hat{\mathbf{b}} &= \left(\frac{\mathbf{b} \cdot \mathbf{a_1}}{\mathbf{a_1} \cdot \mathbf{a_1}}\right) \mathbf{a_1} + \left(\frac{\mathbf{b} \cdot \mathbf{a_2}}{\mathbf{a_2} \cdot \mathbf{a_2}}\right) \mathbf{a_2} \\ &= \left(\frac{8}{4}\right) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{12}{4}\right) \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \\ &= \mathbf{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \mathbf{3} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -1 \\ 1 \\ 5 \end{bmatrix} \end{aligned}$$

**Note:** You *couldn't* use $\hat{\mathbf{b}} = AA^T\mathbf{b}$ because $A$ is not orthogonal (its columns are not orthonormal). However, once you normalize the columns of $A$ to get $A'$, you could also use $\hat{\mathbf{b}} = A'(A')^T$

(b) *(5 points)* Use your answer in $(a)$ to find a least-squares solution to the system $A\mathbf{x} = \mathbf{b}$

We need to find $\widetilde{\mathbf{x}}$ such that $A\widetilde{\mathbf{x}} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is as in $(a)$, so:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \widetilde{\mathbf{x}} = \begin{bmatrix} \widetilde{x} \\ \widetilde{y} \end{bmatrix}, \hat{\mathbf{b}} = \begin{bmatrix} 5 \\ -1 \\ 1 \\ 5 \end{bmatrix}$$

Now row-reduce:

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & -1 & \\ -1 & & \\ -1 & 1 & 1 \\ 1 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 5 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which gives $\boxed{\widetilde{\mathbf{x}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$

**Note:** Another way to do this is to notice that the coefficients of the linear combination in $(a)$ are **2** and **3**. But that corresponds precisely to $\mathbf{x}$ (i.e. $\mathbf{x}$ is the vector of coefficients we need to apply to the columns of $A$ to produce $\mathbf{b}$), hence $\boxed{\widetilde{\mathbf{x}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$

5. *(35 points)* Find a solution to the following wave equation:

$$\begin{cases} u_{tt} = 9u_{xx} & 0 < x < \pi, \quad t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = x^2(\pi - x) & 0 < x < \pi \\ u_t(x,0) = 0 & 0 < x < \pi \end{cases} \tag{1}$$

**Note:** Make sure to show *all* your work, and make sure to do this problem from scratch. Also, at some point, you may have an integral on the denominator. That integral is equal to $\pi$. Finally, be careful!

**Step 1: Separation of variables.** Suppose:

$$u(x,t) = X(x)T(t) \tag{2}$$

Plug (2) into the differential equation (1), and you get:

$$\begin{aligned} (X(x)T(t))_{tt} =& 9\,(X(x)T(t))_{tt} \\ X(x)T''(t) =& 9X''(x)T(t) \end{aligned}$$

Rearrange and get:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{9T(t)} \tag{3}$$

Now $\frac{X''(x)}{X(x)}$ *only* depends on $x$, but by (3) *only* depends on $t$, hence it is constant:

$$\begin{aligned} \frac{X''(x)}{X(x)} =& \lambda \\ X''(x) =& \lambda X(x) \end{aligned} \tag{4}$$

Also, we get:

(5)
$$\begin{aligned}\frac{T''(t)}{9T(t)} =&\lambda \\ T''(t) =&9\lambda T(t)\end{aligned}$$

but we'll only deal with that later (Step 4)

**Step 2:** Consider (4):

$$X''(x) = \lambda X(x)$$

Now use the **boundary conditions** in (1):

$$u_x(0,t) = X'(0)T(t) = 0 \Rightarrow X'(0)T(t) = 0 \Rightarrow X'(0) = 0$$

$$u_x(\pi,t) = X'(\pi)T(t) = 0 \Rightarrow X'(\pi)T(t) = 0 \Rightarrow X'(\pi) = 0$$

Hence we get:

(6)
$$\begin{cases} X''(x) =&\lambda X(x) \\ X'(0) =&0 \\ X'(\pi) =&0 \end{cases}$$

**Step 3: Eigenvalues/Eigenfunctions.** The auxiliary polynomial of (6) is $p(\lambda) = r^2 - \lambda$

Now we need to consider $3$ cases:

<u>Case 1:</u> $\lambda > 0$, then $\lambda = \omega^2$, where $\omega > 0$

Then:

$$r^2 - \lambda = 0 \Rightarrow r^2 - \omega^2 = 0 \Rightarrow r = \pm\omega$$

Therefore:

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$

And

$$X'(x) = A\omega e^{\omega x} - B\omega e^{-\omega x}$$

Now use $X'(0) = 0$ and $X'(\pi) = 0$:

$$X'(0) = A\omega - B\omega = 0 \Rightarrow B\omega = A\omega \Rightarrow A = B \Rightarrow X(x) = Ae^{\omega x} + Ae^{-\omega x}$$

$$X'(\pi) = 0 \Rightarrow A\omega e^{\omega\pi} - Ae^{-\omega\pi} = 0 \Rightarrow Ae^{\omega\pi} = Ae^{-\omega\pi} \Rightarrow e^{\omega\pi} = e^{-\omega\pi} \Rightarrow \omega\pi = -\omega\pi \Rightarrow \omega = 0$$

But this is a **contradiction**, as we want $\omega > 0$.

Case 2: $\lambda = 0$, then $r = 0$, and:

$$X(x) = Ae^{0x} + Bxe^{0x} = A + Bx$$

And:

$$X'(x) = B$$

So:

$$X'(0) = 0 \Rightarrow B = 0 \Rightarrow X(x) = A$$

$$X'(\pi) = 0 \Rightarrow 0 = 0$$

Which is perfectly valid (not a contradiction), so $\boxed{\lambda = 0}$ works and $\boxed{X(x) = A}$

Case 3: $\lambda < 0$, then $\lambda = -\omega^2$, and:

$$r^2 - \lambda = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm\omega i$$

Which gives:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

So:

$$X'(x) = -A\omega\sin(\omega x) + B\omega\cos(\omega x)$$

Again, using $X'(0) = 0$, $X'(\pi) = 0$, we get:

$$X'(0) = B = 0 \Rightarrow \boxed{X(x) = A\cos(\omega x)}, \text{and} X'(x) = -A\omega\sin(\omega x)$$

$$X'(\pi) = -A\omega\sin(\omega\pi) = 0 \Rightarrow \sin(\omega\pi) = 0 \Rightarrow \omega = m, \quad (m = 1, 2, \cdots)$$

This tells us that (combined with Case 2):

(7)
$$\text{Eigenvalues:} \lambda = -\omega^2 = -m^2 \quad (m = 0, 1, 2, \cdots)$$
$$\text{Eigenfunctions:} X(x) = \cos(\omega x) = \cos(mx)$$

**Step 4:** Deal with (5), and remember that $\lambda = -m^2$:

$$T''(t) = 9\lambda T(t)$$
$$\underline{\text{Aux:}}\ r^2 = -9m^2 \Rightarrow r = \pm 3mi \qquad (m = 0, 1, 2, \cdots)$$
$$T(t) = \widetilde{A_m} \cos(3mt) + \widetilde{B_m} \sin(3mt)$$

**Step 5:** Take linear combinations:

(8)
$$u(x,t) = \sum_{m=1}^{\infty} T(t)X(x) = \sum_{m=0}^{\infty} \left( \widetilde{A_m} \cos(3mt) + \widetilde{B_m} \sin(3mt) \right) \cos(mx)$$

**Step 6:** Use the initial condition $u(x, 0) = x^2(\pi - x)$ in (1):

Plug in $t = 0$ in (8), and you get:

(9)
$$u(x,0) = \sum_{m=0}^{\infty} \widetilde{A_m} \cos(mx) = x^2(\pi - x) \qquad \text{on}(0, \pi)$$

Hence we need to find a Fourier cosine series, with $f(x) = x^2(\pi - x) = \pi x^2 - x^3$, so 'evenify' $f$ to get $\widetilde{f}$, and:

$$\begin{aligned}
\widetilde{A_0} =& \frac{\int_{-\pi}^{\pi} \widetilde{f}(x)}{\int_{-\pi}^{\pi} 1} \\
=& \frac{2\int_0^{\pi} \pi x^2 - x^3}{2\pi} \\
=& \frac{1}{\pi} \left[ \pi \frac{x^3}{3} - \frac{x^4}{4} \right]_0^{\pi} \\
=& \frac{1}{\pi} \left( \frac{\pi^4}{3} - \frac{\pi^4}{4} - 0 + 0 \right) \\
=& \frac{\pi^3}{12}
\end{aligned}$$

$$
\begin{aligned}
\widetilde{A_m} =& \frac{\int_{-\pi}^{\pi} \widetilde{f}(x)\cos(mx)}{\int_{-\pi}^{\pi}\cos^2(mx)} \\
=& \frac{2\int_0^{\pi}(\pi x^2 - x^3)\cos(mx)}{\pi} \\
=& \frac{2}{\pi}\left[\left(\pi x^2 - x^3\right)\frac{\sin(mx)}{m} - (2\pi x - 3x^2)\frac{-\cos(mx)}{m^2} + (2\pi - 6x)\frac{-\sin(mx)}{m^3} - (-6)\frac{\cos(mx)}{m^4}\right] \\
=& \frac{2}{\pi}\left(0 + (2\pi^3 - 3\pi^3)\frac{\cos(\pi m)}{m^2} - 0 - 0 + 6\frac{\cos(\pi m) - 1}{m^4}\right) \\
=& \frac{2}{\pi}\left(\frac{-\pi^3(-1)^m}{m^2} + \frac{6((-1)^m - 1)}{m^4}\right) \\
=& \frac{-2\pi^2(-1)^m}{m^2} + \frac{12((-1)^m - 1)}{\pi(m)^4}
\end{aligned}
$$

(for this, we used tabular integration, as well as the fact that the $\sin$ terms are $0$)

**Step 7:** Use the initial condition: $\frac{\partial u}{\partial t}(x,0) = 2\cos(2x) + 8\cos(4x)$ in (1)

First differentiate (8) with respect to $t$:

$$
\frac{\partial u}{\partial t}(x,t) = \sum_{m=1}^{\infty}\left(-3m\widetilde{A_m}\sin(mt) + 3m\widetilde{B_m}\cos(mt)\right)\cos(mx) \tag{10}
$$

Now plug in $t = 0$ in (10):

$$
\frac{\partial u}{\partial t}(x,0) = \sum_{m=1}^{\infty} 3m\widetilde{B_m}\cos(mx) = 0 \tag{11}
$$

By linear independence, all the coefficients are equal to $0$, and hence you get: $\widetilde{B_m} = 0$

**Step 8:** Conclude using (8) and the coefficients $A_m$ and $B_m$ you found:

$$
u(x,t) = \sum_{m=1}^{\infty}\left(\widetilde{A_m}\cos(3mt) + \widetilde{B_m}\sin(3mt)\right)\cos(mx) \tag{12}
$$

where:

$$\widetilde{A_0} = \frac{\pi^3}{12}$$

$$\widetilde{A_m} = \frac{-2\pi^2(-1)^m}{m^2} + \frac{12((-1)^m - 1)}{\pi m^4}$$

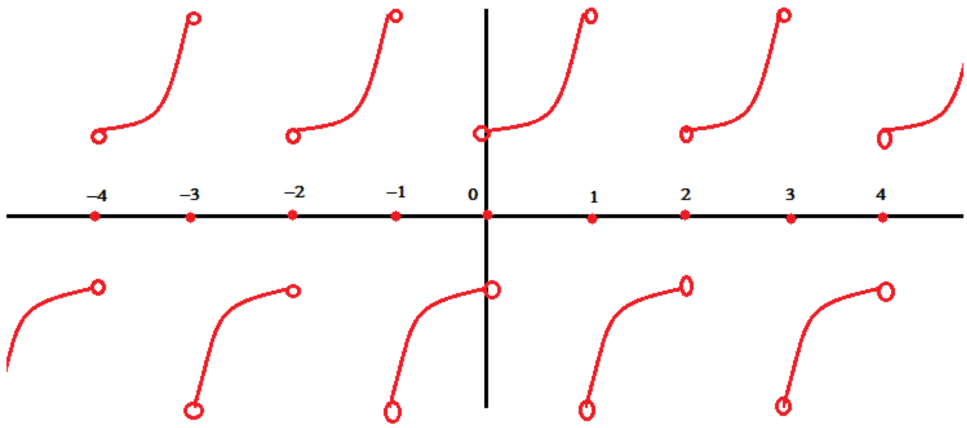
and

$$\widetilde{B_m} = 0$$

6. *(5 points)* Consider $f(x) = x^2 + 1$ on $(0, 1)$.

Draw the graph of $\mathcal{F}(x)$, the Fourier *sine* series of $f$ on $(-4, 4)$
Make sure to label what happens at the endpoints!

For this, just 'oddify' $f$ and repeat the graph of $f$:
54/Math 54 Summer/Exams/Finalgraph.png



7. *(10 points)* Consider $f(x) = \begin{cases} 0 & \text{on}(-1,0) \\ 1 & \text{on}(0,1) \end{cases}$

Parseval's identity states that:

$$\sum_{m=0}^{\infty} (A_m)^2 + (B_m)^2 = \int_{-1}^{1} (f(x))^2$$

Where $A_m$ and $B_m$ are the (full) Fourier coefficients of $f$.

Calculate $A_m$ and $B_m$ and use this to calculate:

$$\sum_{m=1, m\ odd}^{\infty} \frac{1}{m^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} \cdots$$

$$A_0 = \frac{\int_{-1}^{1} f(x)}{\int_{-1}^{1} 1} = \frac{\int_0^1 1}{2} = \frac{1}{2}$$

$$\begin{aligned} A_m = & \frac{\int_{-1}^{1} f(x)\cos(\pi m x)}{\int_{-1}^{1} \cos^2(\pi m x)} \\ = & \frac{\int_0^1 \cos(\pi m x)}{1} \\ = & \left[\frac{\sin(\pi m x)}{\pi m}\right]_0^1 \\ = & 0 \end{aligned}$$

(We used the fact that $f \equiv 0$ on $(-1, 0)$)

$$B_0 = 0$$

$$\begin{aligned}
B_m =& \frac{\int_{-1}^{1} f(x)\sin(\pi m x)}{\int_{-1}^{1}\sin^2(\pi m x)} \\
=& \frac{\int_0^1 \cos(\pi m x)}{1} \\
=& \left[\frac{-\cos(\pi m x)}{\pi m}\right]_0^1 \\
=& \frac{-1}{\pi m}(\cos(\pi m) - 1) \\
=& \frac{-1}{\pi m}((-1)^m - 1)
\end{aligned}$$

(We used the fact that $f \equiv 0$ on $(-1, 0)$)
Now, using Parseval's identity, we get:

$$\begin{aligned}
\sum_{m=0}^{\infty} A_m^2 + B_m^2 =& \int_{-1}^{1} (f(x))^2 \\
A_0^2 + B_0^2 + \sum_{m=1}^{\infty} A_m^2 + B_m^2 =& \int_0^1 1 \\
\left(\frac{1}{2}\right)^2 + 0^2 + \sum_{m=1}^{\infty} 0^2 + \left(\frac{-1}{\pi m}((-1)^m - 1)\right) =& 1 \\
\sum_{m=1}^{\infty} \frac{1}{\pi^2 m^2}((-1)^m - 1)^2 =& 1 - \frac{1}{4} = \frac{3}{4} \\
\sum_{m=1}^{\infty} \frac{((-1)^m - 1)^2}{m^2} =& \frac{3\pi^2}{4}
\end{aligned}$$

And finally, to conclude, notice that $(-1)^m - 1 = 0$ if $m$ is even and $= 2$ if $m$ is odd, hence:

$$\begin{aligned}
\sum_{m=1, m\ odd}^{\infty} \frac{2^2}{m^2} &= \frac{3\pi^2}{4} \\
\sum_{m=1, m\ odd}^{\infty} \frac{1}{m^2} &= \frac{3\pi^2}{4(4)} = \frac{3\pi^2}{16}
\end{aligned}$$

8. *(5 points)* Use the following steps to give an alternate and easier proof of the Cauchy-Schwarz inequality. All the questions are pretty much independent (except for $(d)$)

(a) *(1 point)* What does the Cauchy-Schwarz inequality say?

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

(b) *(1 point)* What is the formula of $\hat{\mathbf{u}}$, the projection of $\mathbf{u}$ on $Span\{\mathbf{v}\}$ ?

$$\hat{\mathbf{u}} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v}} \cdot \mathbf{v}\right) \mathbf{v}$$

(c) *(1 point)* Circle the correct answer:

(A) $\boxed{\|\hat{\mathbf{u}}\| \leq \|\mathbf{u}\|}$

(B) $\|\mathbf{u}\| \leq \|\hat{\mathbf{u}}\|$
(draw a picture)

(d) *(2 points)* Use your formula in $(b)$ and your answer in $(c)$ to solve for $\mathbf{u} \cdot \mathbf{v}$ and (hence) derive the Cauchy-Schwarz inequality!

**Note:** Be careful about when to put $|\cdot|$ or $\|\cdot\|$.

First we use $(c)$, then use $(a)$, and finally take $\mathbf{u} \cdot \mathbf{v}$ outside of $\|\cdot\|$:

$$
\begin{aligned}
\|\hat{\mathbf{u}}\| &\leq \|\mathbf{u}\| \\
\left\| \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v}} \cdot \mathbf{v} \right) \mathbf{v} \right\| &\leq \|\mathbf{u}\| \\
\left| \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v}} \cdot \mathbf{v} \right| \|\mathbf{v}\| &\leq \|\mathbf{u}\| \\
\frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{v}\|^{\cancel{2}}} \cancel{\|\mathbf{v}\|} &\leq \|\mathbf{u}\| \\
\frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{v}\|} &\leq \|\mathbf{u}\| \\
|\mathbf{u} \cdot \mathbf{v}| &\leq \|\mathbf{u}\| \, \|\mathbf{v}\|
\end{aligned}
$$

9. *(3 points)* Suppose $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthonormal. Show that $\mathcal{B}$ is linearly independent!

**Hint:** Use hugging!

**Note:** Let me start the proof for you:

Suppose $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$.

**Goal:** Show that $a = b = c = 0$

First dot the above equation with $\mathbf{u}$ and use orthonormality:

$$\begin{aligned}(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) \cdot \mathbf{u} =& \mathbf{0} \cdot \mathbf{u} = 0 \\ a\mathbf{u} \cdot \mathbf{u} + b\mathbf{v} \cdot \mathbf{u} + c\mathbf{w} \cdot \mathbf{u} =& 0 \\ a(1) + b(0) + c(0) =& 0 \\ a =& 0\end{aligned}$$

Hence $b\mathbf{v} + c\mathbf{w} = \mathbf{0}$. Now dot this with $\mathbf{v}$ and use orthonormality:

$$\begin{aligned}(b\mathbf{v} + c\mathbf{w}) \cdot \mathbf{v} =& \mathbf{0} \cdot \mathbf{v} = 0 \\ b\mathbf{v} \cdot \mathbf{v} + c\mathbf{w} \cdot \mathbf{v} =& 0 \\ b(1) + c(0) =& 0 \\ b =& 0\end{aligned}$$

Hence $c\mathbf{w} = \mathbf{0}$. Finally, dot this with $\mathbf{w}$:

$$\begin{aligned}c\mathbf{w} \cdot \mathbf{w} =& \mathbf{0} \cdot \mathbf{w} = 0 \\ c(1) =& 0 \\ c =& 0\end{aligned}$$

Hence $a = b = c = 0$, and we're done!

**Note:** You were **NOT** allowed to use $a = \frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$. I wrote this on the blackboard!

10. *(2 points)* Who's your favorite Math 54 teacher of all time??? :D

I hope you said ‘Peyam’ or ‘Pie-am’ or $\pi - m$ or any variation thereof :)

**Bonus** *(1 point)* Find the general solution to the following PDE:

$$\begin{cases} u_{xx} + u_{yy} = u \\ u(0,y) = u(1,y) = 0 \end{cases}$$

(where $u = u(x,y)$ and $0 < x < 1, 0 < y < 1$)

Suppose $u(x,y) = X(x)Y(y)$. Then plug this into the above equation:

$$(X(x)Y(y))_{xx} + (X(x)Y(y))_{yy} = X(x)Y(y)$$

$$X''(x)Y(y) + X(x)Y''(y) = X(x)Y(y)$$

And divide all the sides by $X(x)Y(y)$:

$$\begin{aligned} \frac{X''(x)\cancel{Y(y)}}{X(x)\cancel{Y(y)}} + \frac{\cancel{X(x)}Y''(y)}{\cancel{X(x)}Y(y)} &= 1 \\ \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} &= 1 \\ \frac{X''(x)}{X(x)} &= 1 - \frac{Y''(y)}{Y(y)} = \lambda \end{aligned}$$

Hence: $\boxed{X''(x) = \lambda X(x)}$ (and $Y''(y) = (1-\lambda)Y(y)$):

And as usual, we get that $X(0) = 0$ and $X(1) = 0$, and if we do the 3−cases business as usual, we find that: $\boxed{\lambda = -(\pi m)^2 \text{ and } X(x) = \sin(\pi m x)}$ $(m = 1, 2, \cdots)$

Now go back to $Y''(y) = (1-\lambda)Y(y) = (1+(\pi m)^2)Y(y)$. The auxiliary equation is $r^2 = (1 + (\pi m)^2)$, which gives $r = \pm(1 + (\pi m)^2)$, and hence:

$$Y(y) = \widetilde{A_m} e^{(1+(\pi m)^2)y} + \widetilde{B_m} e^{-(1+(\pi m)^2)y}$$

And hence:

$$X(x)Y(y) = \left(\widetilde{A_m} e^{(1+(\pi m)^2)y} + \widetilde{B_m} e^{-(1+(\pi m)^2)y}\right) \sin(\pi m x)$$

And finally take linear combinations:

$$u(x,y)=\sum_{m=1}^{\infty}\left(\widetilde{A_m}e^{(1+(\pi m)^2)y}+\widetilde{B_m}e^{-(1+(\pi m)^2)y}\right)\sin(\pi m x)$$