1. (10 points, 2 points each)

Label the following statements as T or F. Write your answers in the box below!

NOTE: In this question, you do NOT have to show your work! Don’t spend too much time on each question!

(a) **FALSE** If \( \hat{x} \) is the orthogonal projection of \( x \) on \( W \), then \( \hat{x} \) is orthogonal to \( x \).

(Draw a picture)

(b) **FALSE** If \( \hat{u} \) is the orthogonal projection of \( u \) on \( \text{Span} \{v\} \), then:

\[
\hat{u} = \left( \frac{u \cdot v}{v \cdot v} \right) v
\]

(It’s \( \hat{u} = \left( \frac{u \cdot v}{v \cdot v} \right) v \), it has to be a multiple of \( v \))

(c) **TRUE** For any (continuous) \( f \) and \( g \),

\[
\left( \int_0^1 f(t)g(t) dt \right)^2 \leq \left( \int_0^1 (f(t))^2 dt \right) \left( \int_0^1 (g(t))^2 dt \right)
\]

(This is just the Cauchy-Schwarz inequality with \( f \cdot g = \int_0^1 f(t)g(t) dt \):

\[
\left| \int_0^1 f(t)g(t) dt \right| \leq \sqrt{\int_0^1 (f(t))^2 dt} \sqrt{\int_0^1 (g(t))^2 dt}
\]
Now square both sides)

(d) **FALSE** If \( \hat{x} \) is the least-squares solution of \( A\hat{x} = \hat{b} \), then \( \hat{x} \) is the orthogonal projection of \( x \) on \( \text{Col}(A) \).

(We’re not projecting \( x \) onto anything! To find the least-squares solution, project \( b \) onto \( \text{Col}(A) \) to get \( \hat{b} \) and then find \( \hat{x} \) such that \( A\hat{x} = \hat{b} \))

(e) **FALSE** If \( Q \) is an orthogonal matrix, then \( Q \) is invertible. (\( Q \) might not be square!)
2. (10 points) Apply the Gram-Schmidt process to find an orthonormal basis of \( W \), where:

\[
W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \text{Span} \{ u_1, u_2, u_3 \}
\]

**Step 1:** Let \( v_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \)

**Step 2:** Calculate:

\[
\hat{u}_2 = \left( \frac{u_2 \cdot v_1}{v_1 \cdot v_1} \right) v_1 = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}
\]

And let:

\[
v_2 = u_2 - \hat{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}
\]

**Step 3:** Calculate:

\[
u_3 = \left( \frac{u_3 \cdot v_1}{v_1 \cdot v_1} \right) v_1 + \left( \frac{u_3 \cdot v_2}{v_2 \cdot v_2} \right) v_2 = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}
\]

And let:

\[
v_3 = u_3 - \hat{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\]

**Step 4:** Normalize:

\[
w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

**Answer:**

\[
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]
\{w_1, w_2, w_3\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}
3. (10 points) Consider the space \( C[-\frac{\pi}{2}, \frac{\pi}{2}] \) with the dot product:

\[
f \cdot g = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)g(t)dt
\]

Find the orthogonal projection of \( f(t) = \cos(x) \) on

\[W = \text{Span} \{1, \sin(x), \sin(2x)\}\]

And use this to find a function \( g \) which is orthogonal to \( f \).

\[
\hat{f} = \left(\frac{\cos(t) \cdot 1}{1 \cdot 1}\right) + \left(\frac{\cos(t) \cdot \sin(t)}{\sin(t) \cdot \sin(t)}\right) \sin(t) + \left(\frac{\cos(t) \cdot \sin(2t)}{\sin(2t) \cdot \sin(2t)}\right) \sin(2t)
\]

\[
= \left(\frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t) dt}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 dt}\right) + \left(\frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t) \sin(t) dt}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(t) dt}\right) \sin(t) + \left(\frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t) \sin(2t) dt}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(2t) dt}\right) \sin(2t)
\]

\[
= \left(\frac{\sin(t)}{\pi} \right) + 0 \cdot \sin(t) + 0 \cdot \cos(t)
\]

\[
= \frac{2}{\pi}
\]

Here we used the facts that \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t) \sin(t) dt = 0 \) and \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t) \sin(2t) dt = 0 \) which follow from the fact that the integral of an odd function over \((-\frac{\pi}{2}, \frac{\pi}{2})\) is 0.

And finally:

\[
g(t) = f(t) - \hat{f}(t) = \cos(t) - \frac{\pi}{2}
\]
4. (10 points) Consider the (inconsistent) system of equations $Ax = b$, where:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

(a) (5 points) Find the orthogonal projection of $b$ on $Col(A)$

Let $a_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ be the columns of $A$. Then:

$$\hat{b} = \left( \frac{b \cdot a_1}{a_1 \cdot a_1} \right) a_1 + \left( \frac{b \cdot a_2}{a_2 \cdot a_2} \right) a_2$$

$$= \left( \frac{8}{4} \right) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \left( \frac{12}{4} \right) \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -1 \\ 1 \\ 5 \end{bmatrix}$$

Note: You couldn’t use $\hat{b} = AA^Tb$ because $A$ is not orthogonal (its columns are not orthonormal). However, once you normalize the columns of $A$ to get $A'$, you could also use $\hat{b} = A'(A')^T$

(b) (5 points) Use your answer in (a) to find a least-squares solution to the system $Ax = b$
We need to find $\tilde{x}$ such that $A\tilde{x} = \hat{b}$, where $\hat{b}$ is as in (a), so:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 5 \\ -1 \\ 1 \\ 5 \end{bmatrix}$$

Now row-reduce:

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 & 5 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which gives $\tilde{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

**Note:** Another way to do this is to notice that the coefficients of the linear combination in (a) are 2 and 3. But that corresponds precisely to $x$ (i.e. $x$ is the vector of coefficients we need to apply to the columns of $A$ to produce $b$), hence $\tilde{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
5. (35 points) Find a solution to the following wave equation:

$$
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= 9 \frac{\partial^2 u}{\partial x^2} \\
0 &< x < \pi, \quad t > 0 \\
\frac{\partial u}{\partial x}(0, t) &= u_x(\pi, t) = 0 \\
u(x, 0) &= x^2(\pi - x) \\
u_t(x, 0) &= 0
\end{align*}
$$

**Note:** Make sure to show all your work, and make sure to do this problem from scratch. Also, at some point, you may have an integral on the denominator. That integral is equal to $\pi$. Finally, be careful!

**Step 1: Separation of variables.** Suppose:

$$u(x, t) = X(x)T(t)$$

Plug (2) into the differential equation (1), and you get:

$$\begin{align*}
(X(x)T(t))_{tt} &= 9 (X(x)T(t))_{tt} \\
X(x)T''(t) &= 9X''(x)T(t)
\end{align*}$$

Rearrange and get:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{9T(t)}$$

Now $\frac{X''(x)}{X(x)}$ only depends on $x$, but by (3) only depends on $t$, hence it is constant:

$$\frac{X''(x)}{X(x)} = \lambda$$

$$X''(x) = \lambda X(x)$$

Also, we get:
\[ T''(t) = \lambda \]
\[ \frac{T''(t)}{9T(t)} = \lambda \]
\[ T''(t) = 9\lambda T(t) \]

but we’ll only deal with that later (Step 4)

**Step 2:** Consider (4):

\[ X''(x) = \lambda X(x) \]

Now use the **boundary conditions** in (1):

\[ u_x(0, t) = X'(0)T(t) = 0 \Rightarrow X'(0)T(t) = 0 \Rightarrow X'(0) = 0 \]

\[ u_x(\pi, t) = X'(\pi)T(t) = 0 \Rightarrow X'(\pi)T(t) = 0 \Rightarrow X'(\pi) = 0 \]

Hence we get:

\[ \begin{cases} 
X''(x) = \lambda X(x) \\
X'(0) = 0 \\
X'(\pi) = 0 
\end{cases} \]

(6)

**Step 3: Eigenvalues/Eigenfunctions.** The auxiliary polynomial of (6) is \( p(\lambda) = r^2 - \lambda \)

Now we need to consider 3 cases:

**Case 1:** \( \lambda > 0 \), then \( \lambda = \omega^2 \), where \( \omega > 0 \)

Then:

\[ r^2 - \lambda = 0 \Rightarrow r^2 - \omega^2 = 0 \Rightarrow r = \pm \omega \]

Therefore:

\[ X(x) = Ae^{\omega x} + Be^{-\omega x} \]

And

\[ X'(x) = A\omega e^{\omega x} - B\omega e^{-\omega x} \]
Now use $X'(0) = 0$ and $X' (\pi) = 0$:

$$X'(0) = A\omega - B\omega = 0 \Rightarrow B\omega = A\omega \Rightarrow A = B \Rightarrow X(x) = Ae^{\omega x} + Ae^{-\omega x}$$

$$X'(\pi) = A\omega e^{\omega \pi} - Ae^{-\omega \pi} = 0 \Rightarrow A\omega = Ae^{-\omega \pi} \Rightarrow e^{\omega \pi} = e^{-\omega \pi} \Rightarrow \omega \pi = -\omega \pi \Rightarrow \omega = 0$$

But this is a contradiction, as we want $\omega > 0$.

Case 2: $\lambda = 0$, then $r = 0$, and:

$$X(x) = Ae^{0x} + Bxe^{0x} = A + Bx$$

And:

$$X'(x) = B$$

So:

$$X'(0) = 0 \Rightarrow B = 0 \Rightarrow X(x) = A$$

$$X'(\pi) = 0 \Rightarrow 0 = 0$$

Which is perfectly valid (not a contradiction), so $\lambda = 0$ works and $X(x) = A$

Case 3: $\lambda < 0$, then $\lambda = -\omega^2$, and:

$$r^2 - \lambda = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm \omega i$$

Which gives:

$$X(x) = A \cos(\omega x) + B \sin(\omega x)$$

So:

$$X'(x) = -A\omega \sin(\omega x) + B\omega \cos(\omega x)$$

Again, using $X'(0) = 0$, $X'(\pi) = 0$, we get:

$$X'(0) = B = 0 \Rightarrow X(x) = A \cos(\omega x)$$

$$X'(\pi) = -A\omega \sin(\omega \pi) = 0 \Rightarrow \sin(\omega \pi) = 0 \Rightarrow \omega = m, \quad (m = 1, 2, \cdots)$$

This tells us that (combined with Case 2):
(7) Eigenvalues: \( \lambda = -\omega^2 = -m^2 \quad (m = 0, 1, 2, \ldots) \)

Eigenfunctions: \( X(x) = \cos(\omega x) = \cos(mx) \)

**Step 4:** Deal with (5), and remember that \( \lambda = -m^2 \):

\[ T''(t) = 9\lambda T(t) \]

**Aux:** \( r^2 = -9m^2 \Rightarrow r = \pm 3mi \quad (m = 0, 1, 2, \ldots) \)

\[ T(t) = \tilde{A}_m \cos(3mt) + \tilde{B}_m \sin(3mt) \]

**Step 5:** Take linear combinations:

(8) \[ u(x, t) = \sum_{m=1}^{\infty} T(t)X(x) = \sum_{m=0}^{\infty} \left( \tilde{A}_m \cos(3mt) + \tilde{B}_m \sin(3mt) \right) \cos(mx) \]

**Step 6:** Use the initial condition \( u(x, 0) = x^2(\pi - x) \) in (1):

Plug in \( t = 0 \) in (8), and you get:

(9) \[ u(x, 0) = \sum_{m=0}^{\infty} \tilde{A}_m \cos(mx) = x^2(\pi - x) \quad \text{on} \ (0, \pi) \]

Hence we need to find a Fourier cosine series, with \( f(x) = x^2(\pi - x) = \pi x^2 - x^3 \), so ‘evenify’ \( f \) to get \( \tilde{f} \), and:

\[ \tilde{A}_0 = \frac{\int_{-\pi}^{\pi} \tilde{f}(x) \, dx}{\int_{-\pi}^{\pi} 1 \, dx} \]

\[ = \frac{2}{2\pi} \int_{0}^{\pi} \pi x^2 - x^3 \, dx \]

\[ = \frac{1}{\pi} \left[ \pi \left( \frac{x^3}{3} - \frac{x^4}{4} \right) \right]_{0}^{\pi} \]

\[ = \frac{1}{\pi} \left( \pi^3 - \pi^2 - 0 + 0 \right) \]

\[ = \frac{\pi^3}{12} \]
\[ \tilde{A}_m = \frac{\int_{-\pi}^{\pi} \tilde{f}(x) \cos(mx) \, dx}{\int_{-\pi}^{\pi} \cos^2(mx) \, dx} \]
\[ = 2 \int_0^\pi (\pi x^2 - x^3) \cos(mx) \, \frac{\pi}{2} \]
\[ = \frac{2}{\pi} \left[ \left( \pi x^2 - x^3 \right) \frac{\sin(mx)}{m} - (2\pi x - 3x^2) \frac{-\cos(mx)}{m^2} + (2\pi - 6x) \frac{-\sin(mx)}{m^3} - (-6) \frac{\cos(mx)}{m^4} \right] \]
\[ = \frac{2}{\pi} \left( 0 + (2\pi^3 - 3\pi^3) \frac{\cos(\pi m)}{m^2} - 0 - 6 \frac{\cos(\pi m) - 1}{m^4} \right) \]
\[ = \frac{2}{\pi} \left( -\frac{\pi^3(-1)^m}{m^2} + \frac{6((-1)^m - 1)}{m^4} \right) \]
\[ = -2\pi^2(-1)^m \frac{1}{m^2} + \frac{12((-1)^m - 1)}{\pi(m)^4} \]

(for this, we used tabular integration, as well as the fact that the sin terms are 0)

**Step 7:** Use the initial condition: \( \frac{\partial u}{\partial t}(x, 0) = 2 \cos(2x) + 8 \cos(4x) \) in (1)

First differentiate (8) with respect to \( t \):

\[ \frac{\partial u}{\partial t}(x, t) = \sum_{m=1}^{\infty} \left( -3m \tilde{A}_m \sin(mt) + 3m \tilde{B}_m \cos(mt) \right) \cos(mx) \]

Now plug in \( t = 0 \) in (10):

\[ \frac{\partial u}{\partial t}(x, 0) = \sum_{m=1}^{\infty} 3m \tilde{B}_m \cos(mx) = 0 \]

By linear independence, all the coefficients are equal to 0, and hence you get: \( \tilde{B}_m = 0 \)

**Step 8:** Conclude using (8) and the coefficients \( A_m \) and \( B_m \) you found:

\[ u(x, t) = \sum_{m=1}^{\infty} \left( \tilde{A}_m \cos(3mt) + \tilde{B}_m \sin(3mt) \right) \cos(mx) \]

where:
\[ \tilde{A}_0 = \frac{\pi^3}{12} \]

\[ \tilde{A}_m = \frac{-2\pi^2 (-1)^m}{m^2} + \frac{12((-1)^m - 1)}{\pi m^4} \]

and

\[ \tilde{B}_m = 0 \]
6. (5 points) Consider $f(x) = x^2 + 1$ on $(0, 1)$.

Draw the graph of $\mathcal{F}(x)$, the Fourier sine series of $f$ on $(-4, 4)$.
Make sure to label what happens at the endpoints!

For this, just ‘oddify’ $f$ and repeat the graph of $f$:
54/Math 54 Summer/Exams/Finalgraph.png
7. (10 points) Consider \( f(x) = \begin{cases} 0 & \text{on } (-1, 0) \\ 1 & \text{on } (0, 1) \end{cases} \)

Parseval’s identity states that:

\[
\sum_{m=0}^{\infty} (A_m)^2 + (B_m)^2 = \int_{-1}^{1} (f(x))^2
\]

Where \( A_m \) and \( B_m \) are the (full) Fourier coefficients of \( f \).

Calculate \( A_m \) and \( B_m \) and use this to calculate:

\[
\sum_{m=1, m \text{ odd}}^{\infty} \frac{1}{m^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots
\]

\[
A_0 = \frac{\int_{-1}^{1} f(x)}{\int_{-1}^{1} 1} = \frac{\int_{0}^{1} 1}{2} = \frac{1}{2}
\]

\[
A_m = \frac{\int_{-1}^{1} f(x) \cos(\pi mx)}{\int_{-1}^{1} \cos^2(\pi mx)}
\]

\[
= \int_{0}^{1} \cos(\pi mx) \frac{1}{2}
\]

\[
= \left[ \frac{\sin(\pi mx)}{\pi m} \right]_{0}^{1}
\]

\[
= 0
\]

(We used the fact that \( f \equiv 0 \) on \((-1, 0))

\[
B_0 = 0
\]
\[ B_m = \frac{\int_{-1}^{1} f(x) \sin(\pi mx)}{\int_{-1}^{1} \sin^2(\pi mx)} = \frac{\int_{0}^{1} \cos(\pi mx)}{1} = \left[ -\frac{\cos(\pi mx)}{\pi m} \right]_0^1 = \frac{-1}{\pi m} (\cos(\pi m) - 1) = \frac{-1}{\pi m}((-1)^m - 1) \]

(We used the fact that \( f \equiv 0 \) on \((-1, 0))

Now, using Parseval’s identity, we get:

\[
\sum_{m=0}^{\infty} A_m^2 + B_m^2 = \int_{-1}^{1} (f(x))^2 \\

A_0^2 + B_0^2 + \sum_{m=1}^{\infty} A_m^2 + B_m^2 = \int_{0}^{1} 1 \\

\left(\frac{1}{2}\right)^2 + 0^2 + \sum_{m=1}^{\infty} 0^2 + \left(\frac{-1}{\pi m}((-1)^m - 1)\right) = 1 \\

\sum_{m=1}^{\infty} \frac{1}{\pi^2 m^2}((-1)^m - 1)^2 = 1 - \frac{1}{4} = \frac{3}{4} \\

\sum_{m=1}^{\infty} \frac{((-1)^m - 1)^2}{m^2} = \frac{3\pi^2}{4} \\

\]

And finally, to conclude, notice that \((-1)^m - 1 = 0\) if \(m\) is even and \(= 2\) if \(m\) is odd, hence:

\[
\sum_{m=1, m \text{ odd}}^{\infty} \frac{2^2}{m^2} = \frac{3\pi^2}{4} \\

\sum_{m=1, m \text{ odd}}^{\infty} \frac{1}{m^2} = \frac{3\pi^2}{4(4)} = \frac{3\pi^2}{16} 
\]
8. (5 points) Use the following steps to give an alternate and easier proof of the Cauchy-Schwarz inequality. All the questions are pretty much independent (except for (d))

(a) (1 point) What does the Cauchy-Schwarz inequality say?

\[ |\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\| \]

(b) (1 point) What is the formula of \( \hat{\mathbf{u}} \), the projection of \( \mathbf{u} \) on \( \text{Span} \{ \mathbf{v} \} \)?

\[ \hat{\mathbf{u}} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} \]

(c) (1 point) Circle the correct answer:

(A) \( \|\hat{\mathbf{u}}\| \leq \|\mathbf{u}\| \)

(B) \( \|\mathbf{u}\| \leq \|\hat{\mathbf{u}}\| \)

(draw a picture)

(d) (2 points) Use your formula in (b) and your answer in (c) to solve for \( \mathbf{u} \cdot \mathbf{v} \) and (hence) derive the Cauchy-Schwarz inequality!

**Note:** Be careful about when to put \( |\cdot| \) or \( \|\cdot\| \).

First we use (c), then use (a), and finally take \( \mathbf{u} \cdot \mathbf{v} \) outside of \( \|\cdot\| \):
\[\|\hat{u}\| \leq \|u\|\]
\[\left\| \left( \frac{u \cdot v}{v} \right) v \right\| \leq \|u\|\]
\[\left| \frac{u \cdot v}{v} \right| \|v\| \leq \|u\|\]
\[\left| \frac{u \cdot v}{\|v\|^2} \right| \|v\| \leq \|u\|\]
\[\left| \frac{u \cdot v}{\|v\|} \right| \leq \|u\|\]
\[\left| u \cdot v \right| \leq \|u\| \|v\|\]
9. (3 points) Suppose $B = \{u, v, w\}$ is orthonormal. Show that $B$ is linearly independent!

**Hint:** Use hugging!

**Note:** Let me start the proof for you:

Suppose $au + bv + cw = 0$.

**Goal:** Show that $a = b = c = 0$

First dot the above equation with $u$ and use orthonormality:

$$ (au + bv + cw) \cdot u = 0 \cdot u = 0 $$
$$ au \cdot u + bv \cdot u + cw \cdot u = 0 $$
$$ a(1) + b(0) + c(0) = 0 $$
$$ a = 0 $$

Hence $bv + cw = 0$. Now dot this with $v$ and use orthonormality:

$$ (bv + cw) \cdot v = 0 \cdot v = 0 $$
$$ bv \cdot v + cw \cdot v = 0 $$
$$ b(1) + c(0) = 0 $$
$$ b = 0 $$

Hence $cw = 0$. Finally, dot this with $w$:

$$ cw \cdot w = 0 \cdot w = 0 $$
$$ c(1) = 0 $$
$$ c = 0 $$

Hence $a = b = c = 0$, and we’re done!

**Note:** You were NOT allowed to use $a = \frac{\langle v, u \rangle}{u \cdot u}$. I wrote this on the blackboard!

10. (2 points) Who’s your favorite Math 54 teacher of all time?? :D
I hope you said ‘Peyam’ or ‘Pie-am’ or $\pi - m$ or any variation thereof :)
**Bonus (1 point)** Find the general solution to the following PDE:

\[
\begin{cases}
    u_{xx} + u_{yy} = u \\
    u(0, y) = u(1, y) = 0
\end{cases}
\]

(where \( u = u(x, y) \) and \( 0 < x < 1, 0 < y < 1 \))

Suppose \( u(x, y) = X(x)Y(y) \). Then plug this into the above equation:

\[
(X(x)Y(y))_{xx} + (X(x)Y(y))_{yy} = X(x)Y(y)
\]

\[
X''(x)Y(y) + X(x)Y''(y) = X(x)Y(y)
\]

And divide all the sides by \( X(x)Y(y) \):

\[
\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 1
\]

\[
\frac{X''(x)}{X(x)} = 1 - \frac{Y''(y)}{Y(y)} = \lambda
\]

Hence: \( X''(x) = \lambda X(x) \) (and \( Y''(y) = (1 - \lambda)Y(y) \)).

And as usual, we get that \( X(0) = 0 \) and \( X(1) = 0 \), and if we do the 3–cases business as usual, we find that: \( \lambda = - (\pi m)^2 \) and \( X(x) = \sin(\pi mx) \) for \( m = 1, 2, \ldots \)

Now go back to \( Y''(y) = (1 - \lambda)Y(y) = (1 + (\pi m)^2)Y(y) \). The auxiliary equation is \( r^2 = (1 + (\pi m)^2) \), which gives \( r = \pm (1 + (\pi m)^2) \), and hence:

\[
Y(y) = \tilde{A}_m e^{(1 + (\pi m)^2)y} + \tilde{B}_m e^{-(1 + (\pi m)^2)y}
\]

And hence:

\[
X(x)Y(y) = \left( \tilde{A}_m e^{(1 + (\pi m)^2)y} + \tilde{B}_m e^{-(1 + (\pi m)^2)y} \right) \sin(\pi mx)
\]

And finally take linear combinations:
\[ u(x, y) = \sum_{m=1}^{\infty} \left( \tilde{A}_m e^{(1+(\pi m)^2)y} + \tilde{B}_m e^{-(1+(\pi m)^2)y} \right) \sin(\pi mx) \]