

MATH 54 - MOCK MIDTERM 2 - SOLUTIONS

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1. (50 points, 5 pts each)

Label the following statements as **T** or **F**.

Make sure to **JUSTIFY YOUR ANSWERS!!!** You may use any facts from the book or from lecture.

(a) If A and B are square matrices, then $\det(A + B) = \det(A) + \det(B)$.

FALSE

For example, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Then $\det(A) = 1$, $\det(B) = 1$, but $\det(A + B) = \det(O) = 0$ (where O is the zero-matrix).

(b) If $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ are bases for V , and P is the matrix whose i th column is $[\mathbf{d}_i]_{\mathcal{A}}$, then for all \mathbf{x} in V , we have $[\mathbf{x}]_{\mathcal{D}} = P [\mathbf{x}]_{\mathcal{A}}$

FALSE

First of all, $P = \begin{bmatrix} [\mathbf{d}_1]_{\mathcal{A}} & [\mathbf{d}_2]_{\mathcal{A}} & [\mathbf{d}_3]_{\mathcal{A}} \end{bmatrix} = \mathcal{A} \overset{P}{\leftarrow} \mathcal{D}$ (remember, you always evaluate with respect to the new, cool basis, here it is \mathcal{A}), so we should have:

$$[\mathbf{x}]_{\mathcal{A}} = \mathcal{A} \overset{P}{\leftarrow} \mathcal{D} [\mathbf{x}]_{\mathcal{D}} = P [\mathbf{x}]_{\mathcal{D}}$$

And not the opposite!

(c) If $Nul(A) = \{\mathbf{0}\}$, then A is invertible.

FALSE

Don't worry, this got me too! This statement *is* true if A is **SQUARE** ! But if A is not square, this statement is never true!

For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $Nul(A) = \{\mathbf{0}\}$, but A is not invertible, because it is not square.

(d) A 3×3 matrix A with only one eigenvalue cannot be diagonalizable

SUPER FALSE!!!!!!!!!!!!

Remember that to check if a matrix is not diagonalizable, you really have to look at the eigenvectors!

For example, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ has only eigenvalue 2, but is diagonalizable (it's diagonal!). Or you can choose A to be the O matrix, or the identity matrix, this also works!

(e) \mathbb{R}^2 is a subspace of \mathbb{R}^3

FALSE!

\mathbb{R}^2 is not even a *subset* of \mathbb{R}^3 !!! Don't confuse this with $\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$, which *is* a subspace of \mathbb{R}^3 and very similar to \mathbb{R}^2 (but not exactly the same)

- (f) If W is a subspace of V and \mathcal{B} is a basis for V , then some subset of \mathcal{B} is a basis for W .

FALSE

This is also very tricky (this got me too :)), because the ‘opposite’ statement does hold, namely if \mathcal{B} is a basis for W , you can always complete \mathcal{B} to become a basis of V (this is the ‘basis extension theorem’).

As a counterexample, take $V = \mathbb{R}^3$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$,
 and $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (a line in \mathbb{R}^3).

If the statement was true, then one of the vectors in \mathcal{B} would be a basis for W , but this is bogus.

- (g) If \mathbf{v}_1 and \mathbf{v}_2 are 2 eigenvectors of A corresponding to 2 **different** eigenvalues λ_1 and λ_2 , then \mathbf{v}_1 and \mathbf{v}_2 are linearly independent!

TRUE (finally!)

Note: The proof is a bit complicated, but I’ve seen this on a past exam! I think at that point, the professor wanted to get revenge on his students for not coming to lecture!

Remember that eigenvectors have to be nonzero!

Now, assume $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{0}$.

Then apply A to this to get:

$$A(a\mathbf{v}_1 + b\mathbf{v}_2) = A(\mathbf{0}) = \mathbf{0}$$

That is:

$$aA(\mathbf{v}_1) + bA(\mathbf{v}_2) = \mathbf{0}$$

$$a\lambda_1\mathbf{v}_1 + b\lambda_2\mathbf{v}_2 = \mathbf{0}$$

However, we can also multiply the original equation by λ_1 to get:

$$a\lambda_1\mathbf{v}_1 + b\lambda_1\mathbf{v}_2 = \mathbf{0}$$

Subtracting this equation from the one preceding it, we get:

$$b(\lambda_1 - \lambda_2)\mathbf{v}_2 = \mathbf{0}$$

So

$$b(\lambda_1 - \lambda_2) = 0$$

But $\lambda_1 \neq \lambda_2$, so $\lambda_1 - \lambda_2 \neq 0$, hence we get $b = 0$.

But going back to the first equation, we get:

$$a\mathbf{v}_1 = \mathbf{0}$$

So $a = 0$.

Hence $a = b = 0$, and we're done!

- (h) If a matrix A has orthogonal columns, then it is an orthogonal matrix.

FALSE

Remember that an **orthogonal** matrix has to have **orthonormal** columns!

- (i) For every subspace W and every vector \mathbf{y} , $\mathbf{y} - Proj_W \mathbf{y}$ is orthogonal to $Proj_W \mathbf{y}$ (proof by picture is ok here)

TRUE

Draw a picture! $Proj_W \mathbf{y}$ is just another name for \hat{y} .

(j) If \mathbf{y} is already in W , then $Proj_W \mathbf{y} = \mathbf{y}$

TRUE

Again, draw a picture!

If you want a more mathematical proof, here it is:

Let $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_p\}$ be an orthogonal basis for W ($p = \dim(W)$).

$$\text{Then } y = \left(\frac{\mathbf{y} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 + \dots + \left(\frac{\mathbf{y} \cdot \mathbf{w}_p}{\mathbf{w}_p \cdot \mathbf{w}_p} \right) \mathbf{w}_p.$$

But then, by definition of $Proj_W \mathbf{y} = \hat{\mathbf{y}}$, we get:

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 + \dots + \left(\frac{\mathbf{y} \cdot \mathbf{w}_p}{\mathbf{w}_p \cdot \mathbf{w}_p} \right) \mathbf{w}_p = y$$

So $\hat{\mathbf{y}} = \mathbf{y}$ in this case.

2. (20 points) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 7 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 7 \end{bmatrix}$$

Eigenvalues: $\det(A - \lambda I) = 0$ (expanding along last column),
which gives $(\lambda - 1)(\lambda - 7)^2 = 0$, so $\lambda = 1, 7$

$$\underline{\lambda = 1}$$

$$\text{Nul}(A - I) = \text{Nul} \left(\begin{bmatrix} 6 & -6 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix} \right) = \text{Nul} \left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{Span} \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$\underline{\lambda = 7}$$

$$\text{Nul}(A - 7I) = \text{Nul} \left(\begin{bmatrix} 0 & -6 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \right) = \text{Nul} \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Hence:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, P = \begin{bmatrix} -2 & 1 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

3. (20 points) Use the Gram-Schmidt process to obtain an orthonormal basis of $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ -3 \\ 1 \\ 11 \end{bmatrix}$$

First of all, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal basis for W , where:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 = \mathbf{v}_2 - \frac{7}{14} \mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{w}_3 &= \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} \right) \mathbf{w}_2 = \mathbf{v}_3 - \frac{42}{14} \mathbf{w}_1 - \frac{-20}{10} \mathbf{w}_2 \\ &= \mathbf{v}_3 - 3\mathbf{w}_1 + 2\mathbf{w}_2 = \begin{bmatrix} 6 \\ -3 \\ 1 \\ 11 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

Finally, an orthonormal basis for W is $\{\mathbf{w}_1', \mathbf{w}_2', \mathbf{w}_3'\}$, where:

$$\mathbf{w}_1' = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{w}_2' = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \mathbf{w}_3' = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

4. (10 points) Find the determinant of the following matrix A :

$$A = \begin{bmatrix} 1 & 42 & 536 & 789 & 4201 & 123456789 \\ 0 & 1 & 2011 & 2012 & \pi m & \text{Dolphin} \\ 0 & 0 & 2 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2 & -1 \end{bmatrix}$$

Note: The answer may surprise you :)

First of all, expanding along the first column, and then along the first column again, we get that:

$$\det(A) = \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix}$$

Now expanding along the 3rd row (or the second column), we get:

$$\det(A) = - \begin{vmatrix} 2 & 4 & 5 \\ 0 & 3 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

Note: Careful about the signs!

Finally, expanding along the second row (or first column), we get:

$$\det(A) = - \left(3 \begin{vmatrix} 2 & 5 \\ 4 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} \right) = - ((3)(-22) + 12) = 54$$

NO WAY!!! I know, right? I did not expect that at all! :D

Note: Here's a smarter way to evaluate $\det(A)$ (courtesy Rongchang Lei): Just row-reduce!

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & -11 \end{vmatrix} && (R_4 - 2R_1) \\ &= - \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -6 & -11 \end{vmatrix} && (R_2 \leftrightarrow R_3) \\ &= - \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -9 \end{vmatrix} && (R_4 - 2R_3) \\ &= - (2)(1)(3)(-9) && \text{upper triangular matrix} \\ &= 54 \end{aligned}$$

5. (10 points) Find a least squares solution to the following system $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$
$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$$

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Which gives:

$$\hat{\mathbf{x}} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} \\ \frac{1}{9} \end{bmatrix}$$

6. (10 points) Define $T : P_3 \rightarrow P_3$ by:

$$T(p(t)) = tp''(t) - 2p'(t)$$

Find the matrix A of T relative to the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of P_3

First calculate:

- $T(1) = t(0) - 2(0) = 0$
- $T(t) = t(0) - 2(1) = -2$
- $T(t^2) = t(2) - 2(2t) = -2t$
- $T(t^3) = t(6t) - 2(3t^2) = 6t^2 - 6t^2 = 0$

Now evaluate all those vectors with respect to \mathcal{B} :

- $[T(1)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t)]_{\mathcal{B}} = [-2]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t^2)]_{\mathcal{B}} = [-2t]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t^3)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Putting everything together, we get that the matrix of T relative to \mathcal{B} is:

$$A = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

7. (15 points) Let $\mathcal{B} = \left\{ \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$.

(a) Find the change-of-coordinates matrix from \mathcal{B} to \mathcal{C}

We want to find $\mathcal{C} \stackrel{P}{\leftarrow} \mathcal{B}$.

$$\begin{aligned} [\mathcal{C} \mid \mathcal{B}] &\rightarrow \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 1 & 2 & -2 & -1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 0 & -3 & 15 & 6 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 0 & 1 & -5 & -2 \end{array} \right] \\ &\rightarrow \left[\begin{array}{cc|cc} 4 & 0 & 32 & 12 \\ 0 & 1 & -5 & -2 \end{array} \right] \\ &\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{array} \right] \end{aligned}$$

Hence:

$$\mathcal{C} \stackrel{P}{\leftarrow} \mathcal{B} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$$

(b) Find $[\mathbf{x}]_{\mathcal{C}}$, where $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

We have:

$$[\mathbf{x}]_{\mathcal{C}} = \mathcal{C} \stackrel{P}{\leftarrow} \mathcal{B} [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 47 \\ -30 \end{bmatrix}$$

8. (10 points) Find the orthogonal projection of t^2 onto the subspace W spanned by $\{1, t\}$, with respect to the following inner product:

$$\langle p, q \rangle = \int_{-1}^1 p(t)q(t)dt$$

Let $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2$, then:

$$\hat{p}_3 = \left(\frac{\langle p_3, p_1 \rangle}{\langle p_1, p_1 \rangle} \right) p_1 + \left(\frac{\langle p_3, p_2 \rangle}{\langle p_2, p_2 \rangle} \right) p_2 = \left(\frac{\int_{-1}^1 t^2 dt}{\int_{-1}^1 dt} \right) (1) + \left(\frac{\int_{-1}^1 t^3 dt}{\int_{-1}^1 t^2 dt} \right) (t) = \left(\frac{2}{2} \right) (1) + \left(\frac{0}{2} \right) t = \frac{1}{3}$$

$$\hat{p}_3(t) = \frac{1}{3}$$

9. (20 points, 10 points each)

(a) Find a basis for $\text{Row}(A)$ and $\text{Col}(A)$, where:

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 0 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$$

If you row-reduce A , you get that:

$$A \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: You could have row-reduced it further, but no need!

Notice that the pivots are in the all 4 rows and the 1st, 3rd, 4th, and 5th column respectively, hence:

Basis for $\text{Row}(A)$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis for $\text{Col}(A)$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) What is $\text{Rank}(A)$? What is $\text{Dim}(\text{Nul}(A))$?

$$\text{Rank}(A) = 4 \text{ (number of pivots)}$$

$$\text{Dim}(\text{Nul}(A)) = 5 - \text{Rank}(A) = 5 - 4 = 1 \text{ (by Rank-Nullity theorem)}$$

10. (15 points)

(a) Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$, where:

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues:

The characteristic polynomial of A is: $\det(A - \lambda I) = (\lambda - 2)(\lambda + 2) = \lambda^2 - 2\lambda + 2 = 0$ iff $\lambda = 1 \pm i$

Eigenspace for $\lambda = 1 - i$

$$\text{Nul}(A - (1 - i)I) = \text{Nul}\left(\begin{bmatrix} 1 + i & -2 \\ 1 & -1 + i \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 1 & -1 + i \\ 0 & 0 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1 - i \\ 1 \end{bmatrix}\right\}$$

So an eigenvector corresponding to $\lambda = 1 - i$ is $\mathbf{v} = \begin{bmatrix} 1 - i \\ 1 \end{bmatrix}$

Finding P and C :

First of all, for P , we have:

$$\text{Re}(\mathbf{v}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{Im}(\mathbf{v}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence: $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

As for C , we have $Re(\lambda) = 1$, $Im(\lambda) = -1$. Now remember that you put those values on the first **ROW** of C , and you get:

$$C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(Remember that the diagonal entries of C are equal and the nondiagonal ones are opposite of each other)

(b) Write C as a composition of a rotation and a scaling.

C is a rotation by ϕ followed by a scaling r .

r is given by: $r = \sqrt{\det(C)} = \sqrt{2}$, hence:

$$C = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

If you compare this latter matrix with the rotation matrix $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$, you should realize that $\phi = \frac{\pi}{4}$.

Hence C is a rotation by $\boxed{\phi = \frac{\pi}{4}}$ followed by a scaling $\boxed{r = \sqrt{2}}$.