MIDTERM 2 (VOJTA) - ANSWER KEY

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(1) **Note:** The $(2,5)$th entry of A should be a -4, not a 4 (the minus-sign is faint!)

Row-reduce A until you get:

$$\begin{bmatrix} 3 & 18 & 10 & 2 & 7 \\ 0 & 0 & 4 & 11 & 19 \\ 0 & 0 & 0 & 3 & -13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Basis for $Row(A)$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 18 \\ 10 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 11 \\ 19 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ -13 \end{bmatrix} \right\}$$

(b) Basis for $Col(A)$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 10 \\ 2 \\ 8 \\ 0 \\ 12 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ 10 \\ -21 \end{bmatrix} \right\}$$

(2) $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$

Think of this as a change-of-basis problem!

Let P = $[\mathbf{b_1} \quad \mathbf{b_2} \quad \mathbf{b_3}] = [[\mathbf{b_1}]_{\mathcal{E}} \quad [\mathbf{b_2}]_{\mathcal{E}} \quad [\mathbf{b_3}]_{\mathcal{E}}]$

Then: $P = \mathcal{E} \overset{P}{\leftarrow} \mathcal{B}$.

Hence:

$$\mathbf{x} = [\mathbf{x}]_{\mathcal{E}} = \mathcal{E} \overset{P}{\leftarrow} \mathcal{B} \, [\mathbf{x}]_{\mathcal{B}} = P \, [\mathbf{x}]_{\mathcal{B}}$$

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So:

$$[\mathbf{x}]_{\mathcal{B}} = P^{-1}\mathbf{x}$$

(3) $y = 2x - \frac{1}{2}$

In other words, try to solve $A\mathbf{x} = \mathbf{b}$ in the least squares sense, where:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 8 \end{bmatrix}$$

(4) $\mathcal{B} = \left\{ \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \frac{1}{\sqrt{13}} \begin{bmatrix} -2 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$

CAREFUL! You should get $\mathbf{w_3} = \mathbf{0}$, but do **NOT** include it in your basis! What this says is that the firs two vectors are linearly independent, but the third one isn't!

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(6) (a) $\mathcal{B} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(b) 5

(7) This is **HARD**!!!

Suppose $\mathbf{v}$ is an eigenvector of B with eigenvalue λ. Then $B\mathbf{v} = \lambda\mathbf{v}$.
But then:

$$A\mathbf{v} = B^2\mathbf{v} = B(\lambda\mathbf{v}) = \lambda B\mathbf{v} = \lambda\lambda\mathbf{v} = \lambda^2\mathbf{v}$$

So **if** $\mathbf{v}$ is an eigenvector of A, then $\mathbf{v}$ is also an eigenvector of B, but with eigenvalue λ^2.

But if you do the calculations, you find that A has eigenvalues 1 and 4 with corresponding eigenvectors:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Hence, by what we said before, B has the **same** eigenvectors, but with eigenvalues $\sqrt{1} = 1$ and $\sqrt{4} = 2$.

This means that:

$$B = PDP^{-1} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

Where $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Note: The point is: If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$, and this holds for **ANY** k, even fractions! In this problem, you found $B = \sqrt{A} = A^{\frac{1}{2}} = PD^{\frac{1}{2}}P^{-1}$.