MATH 54 – MOCK MIDTERM 1 - SOLUTIONS

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1. *(10 points, 2 pts each)*

Label the following statements as **T** or **F**.

NOTE: The stuff in parentheses is just meant to make you understand why the answer is true or false and is not part of the (official) answer.

(a) If the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ has a row of the form $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$, then the corresponding system has no solutions.

TRUE (this is the fact we talked about in lecture, with $b = 1$)

(b) If A and B are two 2×2 matrices, then $det(AB) = det(BA)$

TRUE $(det(AB) = det(A)det(B) = det(B)det(A) = det(BA))$

(c) The equation $A\mathbf{x} = \mathbf{0}$ always has either one or infinitely many solutions.

TRUE ($\mathbf{x} = \mathbf{0}$ is a solution, and if there is another nontrivial solution $\mathbf{x}$, then $c\mathbf{x}$ is another solution for every c, hence infinitely many solutions. And if not, then the system has only one solution, $\mathbf{x} = \mathbf{0}$)

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(d) If A is a 3×3 matrix with two pivot positions, then the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

TRUE (by the IMT, A is not invertible, from which the second part of the statement follows)

(e) If A, B, C are square matrices with $AB = AC$, then $B = C$

FALSE (take A to be the 0 matrix and B and C any two matrices with $B \neq C$. However, if A is invertible, then the statement is true)

2. *(10 points, 5 points each)* Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer!!!

(a) If A and B are any $n \times n$ matrices, then $(A+B)^{-1} = A^{-1}+B^{-1}$

FALSE Notice, if this is not true for numbers, then it cannot be true for matrices either!

For example, take $A = \begin{bmatrix} 3 \end{bmatrix}$, and $B = \begin{bmatrix} 7 \end{bmatrix}$. Then:

$$(A+B)^{-1} = \left(\begin{bmatrix} 10 \end{bmatrix}\right)^{-1} = \begin{bmatrix} \frac{1}{10} \end{bmatrix}$$

But:

$$A^{-1} + B^{-1} = \begin{bmatrix} \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{1}{7} \end{bmatrix} = \begin{bmatrix} \frac{10}{21} \end{bmatrix} \neq \begin{bmatrix} \frac{1}{10} \end{bmatrix}$$

So $(A+B)^{-1} \neq A^{-1} + B^{-1}$.

NOTE: This whole answer is required for full credit (except maybe the first sentence). That's what I mean by *show* that your counterexample is really a counterexample!

(b) If A (not necessarily square) has a pivot in every row, then the system $A\mathbf{x} = \mathbf{b}$ is always consistent.

TRUE: If A has a pivot in every row, then in the augmented matrix there cannot be a row of the form $\begin{bmatrix} 0 & 0 & 0 \cdots & b \end{bmatrix}$, and hence by the fact discussed in lecture the above system is consistent.

3. *(15 points)* Solve the following system of equations (or say it has no solutions):

$$\begin{cases} 2x + 2y + z = 2 \\ x - y + 3z = 3 \\ 3x + 5y = 1 \end{cases}$$

Write down the augmented matrix and row-reduce:

$$\begin{bmatrix} 2 & 2 & 1 & 2 \\ 1 & -1 & 3 & 3 \\ 3 & 5 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & 3 \\ 2 & 2 & 1 & 2 \\ 3 & 5 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 4 & -5 & -4 \\ 0 & 8 & -9 & -8 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 4 & -5 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence the solution is:

$$\begin{cases} x = 2 \\ y = -1 \\ z = 0 \end{cases}$$

4. *(20 points)* Solve the following system $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 2 & -3 & 9 \\ 2 & -2 & 5 & -9 \\ 1 & -1 & 0 & 3 \\ 4 & 3 & -1 & 8 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -2 \\ -1 \\ 10 \end{bmatrix}$$

Write your answer in (parametric) vector form

$$\begin{bmatrix} 1 & 2 & -3 & 9 & 5 \\ 2 & -2 & 5 & -9 & -2 \\ 1 & -1 & 0 & 3 & -1 \\ 4 & 3 & -1 & 8 & 10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 & 9 & 5 \\ 0 & -6 & 11 & -27 & -12 \\ 0 & 3 & -3 & 6 & 6 \\ 0 & -5 & 11 & -28 & -10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 & 9 & 5 \\ 0 & -6 & 11 & -27 & -12 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & -5 & 11 & -28 & -10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 & 9 & 5 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 5 & -15 & 0 \\ 0 & 0 & 6 & -18 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 & 9 & 5 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -3 & 9 & 5 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now rewrite this as a system:

$$\begin{cases} x = 1 - 2t \\ y = 2 + t \\ z = 3t \\ (t = t) \end{cases}$$

Hence, in vector form, this becomes:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 - 2t \\ 2 + t \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

5. *(15 points)* Calculate AB, where A and B are given, or say that AB is undefined.

(a)

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 4 & 1 & 3 \end{bmatrix}$$

Undefined because A is 3×2 and B is 3×3, and $2 \neq 3$.

(b)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -3 & 1 \\ 0 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 3 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -3 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 7 & 0 \\ 3 & 3 \end{bmatrix}$$

6. *(15 points)* Find A^{-1} (or say 'A is not invertible') where:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

Form the (super) augmented matrix and row-reduce:

$$\begin{aligned}
\begin{bmatrix} A & I \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & -7 & -1 & -3 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -7 & -1 & -3 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 & \frac{1}{2} & -\frac{7}{2} & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{2} & -1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 0 & \frac{3}{2} & -\frac{7}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{2} & -1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{5}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{2} & -1 \end{bmatrix} \\
&= \begin{bmatrix} I & A^{-1} \end{bmatrix}
\end{aligned}$$

Therefore:

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{7}{2} & -1 \end{bmatrix}$$

7. *(15 points)* Find $det(A)$, where:

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 1 \\ 1 & 0 & 0 & 4 \end{bmatrix}$$

$$det(A) = \begin{vmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 1 \\ 1 & 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 4 + 4(1) = 8$$

Note: Here we expanded first along the second row (or column) and then along the third row.