MATH 54 – MOCK MIDTERM 1 – SOLUTIONS

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1. *(15 points)* Solve the following system (or say it has no solutions):

$$\begin{cases} x + y + z = 0 \\ 2x + 2z = 0 \\ 3x + y + 3z = 0 \end{cases}$$

Write down the augmented matrix and row-reduce:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now rewrite this as a system:

$$\begin{cases} x + z = 0 \\ y = 0 \end{cases}$$

That is:

$$\begin{cases} x = -z \\ y = 0 \\ z = z \end{cases}$$

Or in vector form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(where z is free)

Date: Friday, February 20th, 2015.

2. *(20 points)* Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Form the (super) augmented matrix and row-reduce:

$$\begin{aligned}
\begin{bmatrix} A & I \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 & 1 & 0 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} \\
&= \begin{bmatrix} I & A^{-1} \end{bmatrix}
\end{aligned}$$

Hence:

$$A^{-1} = \begin{bmatrix} -1 & -1 & 3 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

3. *(10 points)* What's the next elementary row operation you would use to transform the following matrix in row-echelon form? What is the corresponding elementary matrix?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

Notice that A is almost in row-echelon form, except for that 2 in the last row. Hence, we want to subtract 2 times the second row from the third row to 'eliminate' the 2.

In other words, the answer is: $\boxed{\text{Add } (-2) \text{ times the } 2^{nd} \text{ row to the } 3^{rd} \text{ row}}$.

The corresponding elementary matrix is:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Optional: You can indeed check that:

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

which is indeed in row-echelon form!

4. *(10 points, 5 points each)* Evaluate the following products if they are defined, or say 'undefined'

(a) AB, where:

$$A = \begin{bmatrix} 2 & 5 \\ 0 & 7 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

(b) AB, where:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \\ -2 & 1 & -3 \end{bmatrix}$$

5. *(10 points)* Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation which reflects points in the plane about the origin.

(a) (5 points) Find the matrix A of T.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix}, T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}$$

Hence:

$$A = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$$

(b) (5 points) Use A to find $T(1, 1)$.

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$$

6. *(10 points)* Find the determinant of the following matrix A:

$$A = \begin{bmatrix} 1 & 42 & 536 & 789 & 4201 & 123456789 \\ 0 & 1 & 2011 & 2012 & \pi m & \text{Dolphin} \\ 0 & 0 & 2 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2 & -1 \end{bmatrix}$$

Note: The answer may surprise you :)

First of all, expanding along the first column, and then along the first column again, we get that:

$$det(A) = \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix}$$

Now expanding along the 3rd row (or the second column), we get:

$$det(A) = -\begin{vmatrix} 2 & 4 & 5 \\ 0 & 3 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

Note: Careful about the signs!

Finally, expanding along the second row (or first column), we get:

$$det(A) = -\left(3\begin{vmatrix} 2 & 5 \\ 4 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix}\right) = -\left((3)(-22) + 12\right) = 54$$

NO WAY!!! I know, right? I did not expect that at all! :D

Note: Here's a smarter way to evaluate $det(A)$ (courtesy Rongchang Lei): Just row-reduce!

$$
\begin{aligned}
det(A) &= \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & -11 \end{vmatrix} \qquad (R_4 - 2R_1) \\
&= - \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -6 & -11 \end{vmatrix} \qquad (R_2 \leftrightarrow R_3) \\
&= - \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -9 \end{vmatrix} \qquad (R_4 - 2R_3) \\
&= -\,(2)(1)(3)(-9) \qquad \text{upper triangular matrix} \\
&= 54
\end{aligned}
$$

7. *(15 = 10 + 5 points)*

(a) Find a basis for $Col(A)$, where:

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 0 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$$

If you row-reduce A, you get that:

$$A \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: You could have row-reduced it further, but no need!

Notice that the pivots are in the all 4 rows and the 1st, 3rd, 4th, and 5th column respectively, hence:

<u>Basis for Col(A):</u>

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) What is $dim(Col(A))$?

$dim(Col(A)) = 4$ (= number of pivots)

8. *(10 points)* Find a basis for $Nul(A)$ and $Col(A)$, where A is the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Row-reduce A:

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Notice that every column has a pivot, hence to get a basis for $Col(A)$, go back to A and select all the columns for A, and you get that a basis for $Col(A)$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Finally, to find a basis for $Nul(A)$, we need to solve $A\mathbf{x} = \mathbf{0}$. However, notice that A is a 3×3 matrix with 3 pivots, hence the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. It follows that:

$$Nul(A) = \{\mathbf{0}\}$$

Note: You might be tempted to say that $\{\mathbf{0}\}$ is a basis for $Nul(A)$, but this is technically wrong (but you wouldn't get points off for that). The correct answer is that the basis is $= \emptyset$, but you don't need to know that. The main point is that you should be able to know the procedure for finding $Nul(A)$.