Math 54. Solutions to Sample First Midterm

1. (10 points) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 1 & 1 & 2 \end{bmatrix}$, if it exists. Use the algorithm introduced in Chapter 2.

The algorithm uses row reduction of the matrix $[\,A \quad I\,]$:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 6 & 7 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 2 & 1 & -2 & 1 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -4 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -11 & 3 & 6 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ 0 & 0 & -1 & -4 & 1 & 2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & -5 & 1 & 4 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ 0 & 0 & -1 & -4 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 & 1 & 4 \\ 0 & 1 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 4 & -1 & -2 \end{bmatrix}$$

Therefore the inverse is $\begin{bmatrix} -5 & 1 & 4 \\ -3 & 1 & 1 \\ 4 & -1 & -2 \end{bmatrix}$.

2. (10 points) A matrix A and an echelon form of A are given here:

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & -1 \\ -2 & -4 & 3 & -3 & 0 \\ 1 & 2 & -3 & 3 & 3 \\ 1 & 2 & -2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a). Write the solution set of the homogeneous system $A\vec{x} = \vec{0}$ in parametric vector form (i.e., as a linear combination of fixed vectors, in which the weights are allowed to take on arbitrary values).

Continuing the row reduction gives the following matrix in reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives equations $x_1 = -2x_2 + 3x_5$, $x_3 = x_4 + 2x_5$. The other variables are free, so we have the following solution in parametric vector form:

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} .$$

(b). Give a basis of $\operatorname{Nul} A$.

The above three vectors form a basis:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} .$$

(c). Give a basis of $\operatorname{Col} A$.

The pivot columns are the first and third columns, so use these columns of the *original matrix* A:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 3 \\ -3 \\ -2 \end{bmatrix} .$$

3. (12 points) Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \\ -2 \end{bmatrix}$. Let $H = \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(a). Find a subset of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ that is a basis for H. Explain how you know it is a basis for H.

Row reduce the matrix $A = [\,\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3\,]$:

$$\begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 1 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

In the last matrix, it is easy to see that the third column equals the first column plus twice the second, which therefore also is true of the original matrix: $v_3 = v_1 + 2v_2$. So, the vectors are linearly dependent and do not give a basis.

However, $H = \operatorname{Col} A$, so $\{\vec{v}_1, \vec{v}_2\}$ is a basis for H because those are the pivot columns.

Alternatively, you can use the Spanning Set Theorem in Section 4.3.

(b). Let $\mathcal{B}$ be the basis you found in part (a), and let $\vec{x} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$. Find the $\mathcal{B}$-coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$.

We have $\vec{x} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{v}_1 + \vec{v}_2 + (v_1 + 2v_2) = 2v_1 + 3v_2$, so

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} .$$

4. (8 points) Let A be an $m \times n$ matrix, and let $\vec{b}$ and $\vec{c}$ be vectors in $\mathbb{R}^m$. Assume that both equations $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ are consistent. Explain why the equation $A\vec{x} = \vec{b} + 7\vec{c}$ is consistent.

   Since $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ are consistent, $\vec{b}$ and $\vec{c}$ lie in $\operatorname{Col} A$. Since $\operatorname{Col} A$ is a subspace, $7\vec{c}$ and therefore $\vec{b} + 7\vec{c}$ also lie in $\operatorname{Col} A$. Thus, $A\vec{x} = \vec{b} + 7\vec{c}$ is consistent, because if $\vec{a}_1, \dots, \vec{a}_n$ are the columns of A then

$$\vec{b} + 7\vec{c} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

   for some $x_1, \dots, x_n$, and then $\vec{x} = (x_1, \dots, x_n)$ is a solution of $A\vec{x} = \vec{b} + 7\vec{c}$.

5. (10 points) Use Cramer's Rule to solve for x_2 in the linear system

$$\begin{aligned} 2x_1 \qquad &+ 3x_3 = 2 \\ 3x_1 \qquad &+ 5x_3 = 3 \\ 8x_1 + x_2 \qquad &= 0 \end{aligned}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & 3 & 5 \\ 8 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 0 & 3 \\ 3 & 0 & 5 \\ 8 & 1 & 0 \end{vmatrix}} = \frac{8 \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}}{- \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} = \frac{8(10-9)}{-(10-9)} = \frac{8}{-1} = -8 .$$

   (For the first step, we expanded the numerator about the bottom row and the denominator about the second column.)