MATH 54 – MIDTERM 1 – SOLUTIONS

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1. *(10 points, 2 pts each)*

Label the following statements as **TRUE (T)** or **FALSE (F)**.

(a) **TRUE** If the **augmented** matrix of the system $A\mathbf{x} = \mathbf{b}$ has a pivot in the last column, then the system $A\mathbf{x} = \mathbf{b}$ has no solution.

(that's because there's a row of the form $\begin{bmatrix} 0 & 0 & \cdots & 0 \, b \end{bmatrix}$, where $b \neq 0$)

(b) **FALSE** If A and B are invertible 2×2 matrices, then $(AB)^{-1} = A^{-1}B^{-1}$

(it's $(AB)^{-1} = B^{-1}A^{-1}$, *reverse order*)

(c) **TRUE** If A is a 3×3 matrix such that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^3$.

(the IMT implies that A is invertible, and the IMT again implies the desired result)

(d) **TRUE** The general solution to $A\mathbf{x} = \mathbf{b}$ is of the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_0$, where $\mathbf{x}_p$ is a *particular* solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_0$ is the *general* solution to $A\mathbf{x} = \mathbf{0}$.

(See section 1.5)

(e) **TRUE** If P and D are $n \times n$ matrices, then $det(PDP^{-1}) = det(D)$

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$$det(PDP^{-1}) = det(P)det(D)det(P^{-1}) = \cancel{det(P)}det(D)\frac{1}{\cancel{det(P)}} = det(D)$$

(a)	T
(b)	F
(c)	T
(d)	T
(e)	T

2. *(10 points, 5 points each)* Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer!!!

(a) **FALSE** If A and B are any 2×2 matrices, then $AB = BA$

Take for example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then:

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad BA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

which are not equal to each other!

(in fact, almost any two matrices you chose will give you a counterexample! The most important thing is that you had to find explicit A and B and you had to show that $AB \neq BA$)

(b) **TRUE** The matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ is not invertible.

Notice that the first and the third column of the matrix are equal, hence the columns of A are linearly dependent, so by the IMT A is not invertible!

Note: Many many other answers were possible! For example, you could calculate $det(A) = 0$, or you could row-reduce and say that the matrix has only 2 pivots. Any of those answers is acceptable!

3. *(15 points)* Solve the following system of equations (or say it has no solutions):

$$\begin{cases} 2x + 2y + z = 2 \\ 3x + 4y + 2z = 3 \\ x + 2y - z = -3 \end{cases}$$

Write down the augmented matrix and row-reduce:

$$
\begin{aligned}
&\begin{bmatrix} 2 & 2 & 1 & 2 \\ 3 & 4 & 2 & 3 \\ 1 & 2 & -1 & -3 \end{bmatrix} \\
\to &\begin{bmatrix} 1 & 2 & -1 & -3 \\ 2 & 2 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{bmatrix} \\
\to &\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & -2 & 3 & 8 \\ 0 & -2 & 5 & 12 \end{bmatrix} \\
\to &\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & -2 & 3 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \\
\to &\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & -2 & 3 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
\to &\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
\to &\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
\to &\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}
\end{aligned}
$$

Hence the solution is:

$$
\begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}
$$

4. *(20 points)* Solve the following system $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 2 & 3 & 1 & -6 \\ -1 & 2 & -4 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix}$$

Write your answer in (parametric) vector form

$$\begin{bmatrix} 1 & 1 & 1 & -3 & 3 \\ 2 & 3 & 1 & -6 & 8 \\ -1 & 2 & -4 & 3 & 3 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 1 & -3 & 3 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 3 & -3 & 0 & 6 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 1 & -3 & 3 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 2 & -3 & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now rewrite this as a system (**careful about the variables!**):

$$\begin{cases} x + 2z - 3t = 1 \\ y - z = 2 \\ (z = z) \\ (t = t) \end{cases}$$

$$\begin{cases} x = 1 - 2z + 3t \\ y = 2 + z \\ (z = z) \\ (t = t) \end{cases}$$

Hence, in vector form, this becomes:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1-2z+3t \\ 2+z \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2z \\ z \\ z \\ 0 \end{bmatrix} + \begin{bmatrix} 3t \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

5. *(15 points, 5 points each)*

(a) Calculate AB, or say that AB is undefined.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

This **is** defined, and AB is a 3×3 matrix:

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(b) Calculate AB, or say that AB is undefined.

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 3 & 0 \end{bmatrix}$$

AB is **undefined** because A is 3×1 and B is 3×2, and $1 \neq 3$.

(c) Calculate A^2, where:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note: If you're smart about this, you recognize A as the matrix which interchanges the 2 rows of a 2×2 matrix, so applying A

twice should just give you the identity matrix (i.e. the matrix that does 'nothing')!

6. *(15 points)* Find A^{-1} (or say 'A is not invertible') where:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

Form the (super) augmented matrix and row-reduce:

$$\begin{aligned}
\begin{bmatrix} A & I \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & -4 & -7 & -2 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 7 & -2 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -3 & 7 & -2 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{bmatrix} \\
&= \begin{bmatrix} I & A^{-1} \end{bmatrix}
\end{aligned}$$

Hence

$$A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & -2 \\ 2 & -4 & 1 \end{bmatrix}$$

7. *(15 points)* Find $det(A)$, where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 2 & 0 & 4 & 0 & 5 \\ 1 & 2 & 5 & -2 & 0 \\ 2 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

First expand along the second column (**be careful about the sign!**)

$$det(A) = -2 \begin{vmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 0 & 5 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{vmatrix}$$

Then expand along the third column:

$$det(A) = (-2)(3) \begin{vmatrix} 2 & 4 & 5 \\ 2 & 3 & 1 \\ 0 & 1 & -1 \end{vmatrix} = (-6) \begin{vmatrix} 2 & 4 & 5 \\ 2 & 3 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

Now expand along the last row:

$$det(A) = (-6) \left((-1) \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} \right) == (-6)(8+2) = -60$$

So $\boxed{det(A) = -60}$

Bonus *(3 points)* Find $det(A)$, where:

$$A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & t & t^2 & t^3 \end{bmatrix}$$

The trick is to **row-reduce** A (but you have to be **careful about the order!**

First, add (-1) times the first row to the second, third, and fourth rows while keeping the first row fixed (remember that this doesn't change the determinant):

$$det(A) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & y-x & y^2-x^2 & y^3-x^3 \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & t-x & t^2-x^2 & t^3-x^3 \end{vmatrix}$$

Now notice that $y^2 - x^2 = (y-x)(y+x)$, and $y^3 - x^3 = (y-x)(y^2+xy+x^2)$, and so you can 'factor' out $(y-x)$ from the second row:

$$det(A) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & (y-x) & (y-x)(y+x) & (y-x)(y^2+yx+x^2) \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & t-x & t^2-x^2 & t^3-x^3 \end{vmatrix}$$

$$=(y-x)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+yx+x^2 \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & t-x & t^2-x^2 & t^3-x^3 \end{vmatrix}$$

But you can apply the exact same reasoning to the third and the fourth row, to get:

$$det(A) = (y-x)(z-x)(t-x)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 1 & z+x & z^2+xz+x^2 \\ 0 & 1 & t+x & t^2+xt+x^2 \end{vmatrix}$$

But now, add (-1) times the second row to the third row and the fourth row (all while leaving the second row fixed), to get:

$$det(A) = (y-x)(z-x)(t-x)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & z-y & z^2-y^2+xz-xy \\ 0 & 0 & t-y & t^2-y^2+xt-xy \end{vmatrix}$$

But $z^2-y^2+xz-xy = (z-y)(z+y)+(z-y)x = (z-y)(z+y+x) = (z-y)(x+y+z)$, so you can factor out $(z-y)$ from the third row:

$$det(A) = (y-x)(z-x)(t-x)(z-y)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & 1 & x+y+z \\ 0 & 0 & t-y & t^2-y^2+xt-xy \end{vmatrix}$$

Similarly, you can factor out $(t-y)$ from the fourth row:

$$det(A) = (y-x)(z-x)(t-x)(z-y)(t-y)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & 1 & x+y+z \\ 0 & 0 & 1 & x+y+t \end{vmatrix}$$

Finally, add (-1) times the third row to the fourth row:

$$det(A) = (y-x)(z-x)(t-x)(z-y)(t-y)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & 1 & x+y+z \\ 0 & 0 & 0 & t-z \end{vmatrix}$$

But this last matrix is upper-triangular, hence its determinant is $(1)(1)(1)(t-z)$, and we finally get:

$$\boxed{det(A) = (y-x)(z-x)(t-x)(z-y)(t-y)(t-z)}$$

The way to read this is as follows:

First fix x (the first variable), then take products of differences of the other variables with x, i.e. $(y-x)(z-x)(t-x)$.

Then fix y (the second variable), and take products of differences of all the other variables (except for x) with y, i.e. $(z-y)(t-y)$.

Finally, fix z (the next-to-last variable), and take products of differences of all the other variables (except for x and y) with z, i.e. $(t-z)$.

And then take the product of everything you found to get:

$$det(A) = (y-x)(z-x)(t-x)(z-y)(t-y)(t-z)$$

In fact, there's a (natural) generalization of this! Google 'Vandermonde matrix' to learn more about this!