SOLUTIONS

1. *(20 points, 2 points each)* Label each statement as **TRUE** or **FALSE**. In this question, you do **NOT** have to justify your answer. Each correct answer will get 2 points and each incorrect or illegible answer will get 0 points.

F (a) If A is a 4×6 matrix with 2 pivot columns, then $Nul(A) = \mathbb{R}^4$. $\longrightarrow$ NUL(A) IS A 4 DIM SUBSPACE OF $\mathbb{R}^6$

T (b) If $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$, then the columns of A must span $\mathbb{R}^m$ $\longrightarrow$ BY DEFINITION / ROW THEOREM

T (c) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a one-to-one linear transformation, then T must also be onto $\mathbb{R}^3$ $\longrightarrow$ BY IMT (THE MATRIX A OF T IS INVERTIBLE)

T (d) A matrix A such that $\det(A^2) - 2\det(A) + \det(I) = 0$ must be invertible. $\longrightarrow$ $(\text{DET}(A))^2 - 2\,\text{DET}(A) + 1 = 0 \Rightarrow (\text{DET}(A)-1)^2 = 0 \Rightarrow \text{DET}(A) = 1 \neq 0$

F (e) For any $\mathbf{b}$ in $\mathbb{R}^m$, the set of solutions to $A\mathbf{x} = \mathbf{b}$ is a subspace of $\mathbb{R}^n$. $\longrightarrow$ ONLY IF $b = 0$. IF $b \neq 0$, THEN 0 IS NOT IN IT!

F (f) If A does not have a pivot in every row, then $A\mathbf{x} = \mathbf{b}$ must be inconsistent for every $\mathbf{b}$. $\longrightarrow$ FOR <u>SOME</u> b

F (g) If A and B are row-equivalent, then $Col(A) = Col(B)$ $\begin{bmatrix}1&1\\1&1\end{bmatrix} \sim \begin{bmatrix}1&1\\0&0\end{bmatrix}$ BUT $\text{SPAN}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} \neq \text{SPAN}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

T (h) The matrix $\begin{bmatrix}1&2&3\\2&4&6\\3&6&9\end{bmatrix}$ is not invertible. $\longrightarrow$ COLUMNS ARE LIN DEP

F (i) If $AB = I$, then A must be invertible $\longrightarrow$ $\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I$

F (j) If $A = \begin{bmatrix}1&0\\0&0\end{bmatrix}$, then $Nul(A) = Span\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ $\longrightarrow$ $\text{SPAN}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$

2. *(15 points, 5 points each)* In this question, you do **NOT** need to justify your answers, but there will be no partial credit for each subpart.

Find all the values of h and k (if any) for which $A\mathbf{x} = \mathbf{b}$ has

(a) No solutions
(b) Exactly one solution
(c) Infinitely many solutions

$$A = \begin{bmatrix} 1 & -2 \\ 2 & h \\ 3 & -6 \\ -1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ k \\ -3 \end{bmatrix}$$

$$\begin{array}{c} (\times -2) \\ (\times -3) \\ (\times 1) \end{array} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & h & 6 \\ 3 & -6 & K \\ -1 & 2 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & h+4 & 0 \\ 0 & 0 & K-9 \\ 0 & 0 & 0 \end{array}\right]$$

(a) $K \neq 9$ AND ANY h

(INCONSISTENT $\Leftrightarrow$ THERE IS A ROW OF THE FORM $[0 \ 0 \mid *]$, $* \neq 0$ $\Leftrightarrow$ $K-9=0$ $\Leftrightarrow$ $K=9$)

(b) $K=9$ AND $h \neq -4$

(WANT ≥ 1 SOL $\Rightarrow$ $\underline{K=9}$ BY (a), AND WANT NO FREE VARIABLES $\Leftrightarrow$ PIVOT IN EVERY COLUMN $\Leftrightarrow$ $h+4 \neq 0$ $\Leftrightarrow$ $\underline{h \neq -4}$

(c) $K=9$ AND $h=-4$

(WANT ≥ 1 SOL $\Leftrightarrow$ $\underline{K=9}$ BY (a), AND WANT FREE VARIABLES $\Leftrightarrow$ $h+4=0$ $\Leftrightarrow$ $\underline{h=-4}$)

3. *(10 points)* Calculate the following determinant.

$$\begin{vmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 5 & 0 & 8 & 0 \\ 3 & 9 & 0 & 1 & 0 & 0 & 1 \\ 4 & 5 & 7 & 0 & 3 & 5 & 7 \\ 8 & 3 & 0 & 1 & 0 & 1 & 2 \\ 3 & 2 & 0 & 1 & 0 & 2 & 1 \end{vmatrix}$$

$$= (2)(1)\begin{vmatrix} 4 & 5 & 0 & 8 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 7 & 0 & 3 & 5 & 7 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 1 \end{vmatrix}$$

$$= (2)(3)\begin{vmatrix} 4 & 5 & 8 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{vmatrix}$$

$$= (6)(4)\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 24\left(1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right)$$

$$= 24(\cancel{1} - 4 + 2 - \cancel{1})$$

$$= -48$$

4. *(10 points)* Solve the system $A\mathbf{x} = \mathbf{b}$

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ (\div -4) \end{array}\left[\begin{array}{ccc|c} 2 & 1 & -3 & 6 \\ 1 & 2 & 4 & -1 \\ -4 & 4 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 2 & 1 & -3 & 6 \\ 1 & -1 & 0 & 0 \end{array}\right] (\times -2)\ (\times -1)$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 0 & -3 & -11 & 8 \\ 0 & -3 & -4 & 1 \end{array}\right] (\times -1)$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 0 & -3 & -11 & 8 \\ 0 & 0 & 7 & -7 \end{array}\right] (\div 7)$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 0 & -3 & -11 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right] (\times 11)\ (\times -4)$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] (\div -3)$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] (\times -2)$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\boxed{\text{ANS} \quad \underline{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$$

5. *(20 = 10 + 1 + 4 + 2 + 3 points)* For the following matrix A, find:

(a) A basis for $Nul(A)$
(b) $\dim(Nul(A))$
(c) A basis for $Col(A)$
(d) $Rank(A)$
(e) State the Rank Theorem

(a)

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 \\ 1 & 2 & -4 & 10 & 13 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -3 & 1 & -5 & -7 \\ 1 & -2 & 0 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 7 & 9 \\ 0 & 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} (x-3) \\ (x-7) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 0 & 16 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (x-1)$$

$$\sim \begin{array}{c} \begin{matrix} & & z & & s \end{matrix} \\ \begin{bmatrix} 1 & 0 & -2 & 0 & 9 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{cases} x - 2z + 9s = 0 \\ y - z + 7s = 0 \\ t - s = 0 \end{cases} \Longrightarrow \begin{cases} x = 2z - 9s \\ y = z - 7s \\ t = s \end{cases}$$

$$\Rightarrow \underline{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \end{bmatrix} = \begin{bmatrix} 2z-9s \\ z-7s \\ z \\ s \\ s \end{bmatrix} = \begin{bmatrix} 2z \\ z \\ z \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -9s \\ -7s \\ 0 \\ s \\ s \end{bmatrix}$$

$$= z \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \end{bmatrix} \Longrightarrow \underline{\text{BASIS FOR } NUL(A)} \quad \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(b) $DIM(NUL(A)) = 2$

(c) PIVOTS IN COLUMNS 1, 2, 4 : $\underline{\text{BASIS FOR } COL(A)}$ $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{bmatrix} \right\}$

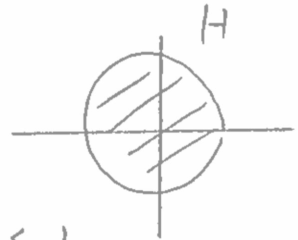
(d) $RANK(A) = 3$, (e) $DIM(NUL(A)) + RANK(A) = N$

6. *(10 points, 5 points each)* Label each statement as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer, meaning that if the statement is true, you have to explain why it's true, and if the statement is false, you have to give an explicit counterexample and show why it's a counterexample.

FALSE

(a) The following set is a subspace of $\mathbb{R}^2$:

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 \text{ such that } x^2 + y^2 \leq 1 \right\}$$



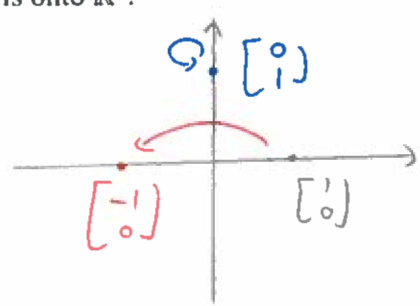
$U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ IS IN H BECAUSE $1^2 + 0^2 = 1 \leq 1$

$V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ IS IN H BECAUSE $0^2 + 1^2 = 1 \leq 1$

BUT $U + V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ IS NOT IN H BECAUSE $1^2 + 1^2 = 2 > 1$

TRUE

(b) The linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ which reflects points in the plane about the y-axis is onto $\mathbb{R}^2$.



FIND A (MATRIX OF T)

$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

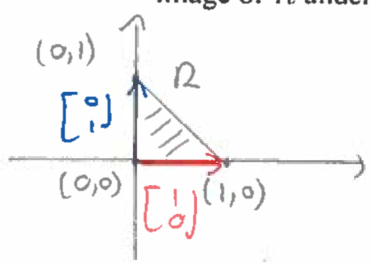
$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

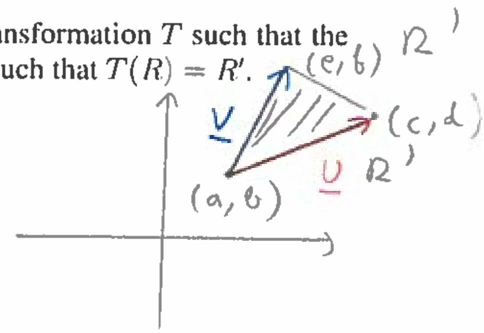
SINCE A HAS A PIVOT IN EVERY ROW, $T(\underline{x}) = A\underline{x}$ IS ONTO $\mathbb{R}^2$

(OR: SINCE A IS INVERTIBLE, BY IMT, $T(\underline{x}) = A\underline{x}$ IS ONTO $\mathbb{R}^2$)

7. *(15 points, 5 points each)* Let R be the triangle in $\mathbb{R}^2$ with vertices $(0,0),(1,0),(0,1)$ and R' be the triangle in $\mathbb{R}^2$ with vertices $(a,b),(c,d),(e,f)$.

(a) Find the matrix A of the linear transformation T such that the image of R under T is R', that is such that $T(R) = R'$.





$$T\begin{bmatrix}1\\0\end{bmatrix} = \underline{u} = \begin{bmatrix}c-a\\d-b\end{bmatrix}, \quad T\begin{bmatrix}0\\1\end{bmatrix} = \underline{v} = \begin{bmatrix}e-a\\f-b\end{bmatrix}, \quad \boxed{A = \begin{bmatrix}c-a & e-a\\ d-b & f-b\end{bmatrix}}$$

(b) Using the formula of determinants in terms of areas, calculate the area of R'. No need to expand out the expression you found.

$$\text{AREA}(R') = |\text{DET}(A)|\ \text{AREA}(R)$$

$$= \left|\text{DET}\begin{bmatrix}c-a & e-a\\ d-b & f-b\end{bmatrix}\right| \frac{1}{2}(1)(1)$$

← R = TRIANGLE W/ BASE 1 AND HEIGHT 1

$$= \frac{1}{2}\left|(c-a)(f-b)-(e-a)(d-b)\right|$$

(c) [redacted], show that your answer in (b) can be written as the absolute value of

$$\frac{1}{2}\begin{vmatrix}1 & a & b\\ 1 & c & d\\ 1 & e & f\end{vmatrix}$$

(x−1) (x−1)

$$= \frac{1}{2}\begin{vmatrix}1 & a & b\\ 0 & c-a & d-b\\ 0 & e-a & f-b\end{vmatrix} = \frac{1}{2}\begin{vmatrix}c-a & d-b\\ e-a & f-b\end{vmatrix}$$

$$= \frac{1}{2}\left((c-a)(f-b)-(e-a)(d-b)\right)$$

$$\underline{\text{ANS}} = \frac{1}{2}\left|(c-a)(f-b)-(e-a)(d-b)\right| = \text{ANSWER IN (b)!}$$