MATH 54 – QUIZ 2 – SOLUTIONS

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1. *(4 points)* Determine for which c the following vectors are linearly independent:

$$\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ c \\ 3 \end{bmatrix}$$

All we need to do is to row-reduce the matrix:

$$\begin{bmatrix} 3 & -6 & 9 & 0 \\ -6 & 4 & c & 0 \\ 1 & -3 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 18+c & 0 \end{bmatrix}$$

In order for the vectors to be linearly independent, by the Useful Test 2 (covered in section on Tuesday), the number of pivots needs to be equal to the number of columns of A (otherwise the equation $A\mathbf{x} = \mathbf{0}$ has a free variable), which is 3. Therefore we need 3 pivots, which means we need $18 + c \neq 0$, so $\boxed{c \neq -18}$.

2. *(6 points total)* Suppose $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is a linear transformation such that:

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad T\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+4c \\ 2+2c \end{bmatrix}$$

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(a) *(4 points)* Find the matrix of T

The only thing that's left to figure out is $T\begin{bmatrix}0\\0\\1\end{bmatrix}$.

Now notice that because T is linear, we have:

$$T\begin{bmatrix}2\\-1\\2\end{bmatrix} = 2T\begin{bmatrix}1\\0\\0\end{bmatrix} + (-1)T\begin{bmatrix}0\\1\\0\end{bmatrix} + 2T\begin{bmatrix}0\\0\\1\end{bmatrix} = 2\begin{bmatrix}2\\4\end{bmatrix} + (-1)\begin{bmatrix}3\\6\end{bmatrix} + 2T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} + 2T\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Hence:

$$\begin{bmatrix}1\\2\end{bmatrix} + 2T\begin{bmatrix}0\\0\\1\end{bmatrix} = T\begin{bmatrix}2\\-1\\2\end{bmatrix} = \begin{bmatrix}1+4c\\2+2c\end{bmatrix}$$

And so, solving for $T\begin{bmatrix}0\\0\\1\end{bmatrix}$, we get:

$$T\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}1+4c\\2+2c\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}2c\\c\end{bmatrix}$$

And therefore, the matrix of T is:

$$A = \begin{bmatrix} T\begin{bmatrix}1\\0\\0\end{bmatrix} & T\begin{bmatrix}0\\1\\0\end{bmatrix} & T\begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}2 & 3 & 2c\\4 & 6 & c\end{bmatrix}$$

(b) *(1 point)* For which c is T one-to-one? Justify **briefly**

Row-reducing A, we get:

$$A = \begin{bmatrix}2 & 3 & 2c\\4 & 6 & c\end{bmatrix} \longrightarrow \begin{bmatrix}2 & 3 & 2c\\0 & 0 & c\end{bmatrix}$$

In order for T to be one-to-one, we need to have as many pivots in A as **columns** (otherwise the equation $A\mathbf{x} = \mathbf{0}$ has a free variable), so we need 3 pivots, but this is impossible as there can only be 2 pivots at most. Hence T is $\boxed{\text{never}}$ one-to-one

(c) *(1 point)* For which c is T onto $\mathbb{R}^2$? Justify **briefly**

In order for T to be onto $\mathbb{R}^2$, we need to have as many pivots in A as **rows** (otherwise the equation $A\mathbf{x} = \mathbf{b}$ isn't always consistent), so we need 2 pivots, which means that we need to have $\boxed{c \neq 0}$