# MATH 54 – QUIZ 3 – SOLUTIONS

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1. *(4 points)* Find the inverse of the following matrix (or say it's not invertible)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

Form the big matrix:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

And row-reduce until you get that the left-hand-side becomes the identity:

$$\begin{bmatrix} I & A^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 & -1 & -2 & 2 \end{bmatrix}$$

**Note:** From now on, for sake of clarity, I'm skipping the row-reduction process. But if you have any questions, please don't hesitate to e-mail me!

And therefore, we get:

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 3 & -1 & -2 & 2 \end{bmatrix}$$

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2. *(2 points)* Recall that a matrix $A$ is symmetric if $A^T = A$. Show that $AB + B^T A^T$ is always symmetric (even if $A$ and $B$ are not necessarily symmetric).

All we need to do is calculate $\left(AB + B^T A^T\right)^T$ and show that it's equal to $AB + B^T A^T$. Using property of transposes, we get:

$$\begin{aligned}\left(AB + B^T A^T\right)^T =&(AB)^T + \left(B^T A^T\right)\\ =&B^T A^T + \left(A^T\right)^T \left(B^T\right)^T\\ =&B^T A^T + AB\\ =&AB + B^T A^T\end{aligned}$$

Hence $AB + B^T A^T$ is symmetric.

3. *(4 points)* Find the determinant of the following matrix, where $x, y, z, t$ are distinct real numbers. Write your answer in factored form.

$$A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & t & t^2 & t^3 \end{bmatrix}$$

**Hint:** You might find the formula $p^3 - q^3 = (p-q)(p^2+pq+q^2)$ (where $p$ and $q$ are real numbers) useful!

The trick is to **row-reduce** $A$ (but you have to be **careful about the order!**

First, add $(-1)$ times the first row to the second, third, and fourth rows while keeping the first row fixed (remember that this doesn't change the determinant):

$$det(A) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & y-x & y^2-x^2 & y^3-x^3 \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & t-x & t^2-x^2 & t^3-x^3 \end{vmatrix}$$

Now notice that $y^2 - x^2 = (y-x)(y+x)$, and $y^3 - x^3 = (y-x)(y^2+xy+x^2)$, and so you can 'factor' out $(y-x)$ from the second row:

$$det(A) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & (y-x) & (y-x)(y+x) & (y-x)(y^2+yx+x^2) \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & t-x & t^2-x^2 & t^3-x^3 \end{vmatrix}$$

$$=(y-x) \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+yx+x^2 \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & t-x & t^2-x^2 & t^3-x^3 \end{vmatrix}$$

But you can apply the exact same reasoning to the third and the fourth row, to get:

$$det(A) = (y-x)(z-x)(t-x) \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 1 & z+x & z^2+xz+x^2 \\ 0 & 1 & t+x & t^2+xt+x^2 \end{vmatrix}$$

But now, add $(-1)$ times the second row to the third row and the fourth row (all while leaving the second row fixed), to get:

$$det(A) = (y-x)(z-x)(t-x) \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & z-y & z^2-y^2+xz-xy \\ 0 & 0 & t-y & t^2-y^2+xt-xy \end{vmatrix}$$

But $z^2-y^2+xz-xy = (z-y)(z+y)+(z-y)x = (z-y)(z+y+x) = (z-y)(x+y+z)$, so you can factor out $(z-y)$ from the third row:

$$det(A) = (y-x)(z-x)(t-x)(z-y) \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & 1 & x+y+z \\ 0 & 0 & t-y & t^2-y^2+xt-xy \end{vmatrix}$$

Similarly, you can factor out $(t-y)$ from the fourth row:

$$det(A) = (y-x)(z-x)(t-x)(z-y)(t-y)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & 1 & x+y+z \\ 0 & 0 & 1 & x+y+t \end{vmatrix}$$

Finally, add $(-1)$ times the third row to the fourth row:

$$det(A) = (y-x)(z-x)(t-x)(z-y)(t-y)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & 1 & x+y+z \\ 0 & 0 & 0 & t-z \end{vmatrix}$$

But this last matrix is upper-triangular, hence its determinant is $(1)(1)(1)(t-z)$, and we finally get:

$$\boxed{det(A) = (y-x)(z-x)(t-x)(z-y)(t-y)(t-z)}$$