MATH 54 – QUIZ 4 – SOLUTIONS

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1. *(4 points)* Recall that a matrix A is symmetric if and only if $A^T = A$. Does the set of $n \times n$ symmetric matrices (with real entries) form a vector space? Justify **carefully**.

Let W be the set of $n \times n$ symmetric matrices. We want to show that W is a subspace of $V = M_{n \times n}$, the set of all $n \times n$ matrices (with real entries)

All we need to show that the zero-matrix is in W, and that W is closed under addition and scalar multiplication.

Zero-vector: The $n \times n$ zero-matrix O satisfies $O^T = O$, therefore O is in W

Closed under addition: Suppose A and B are in W. Then $A^T = A$ and $B^T = B$. But then, by properties of transpose:

$$(A + B)^T = A^T + B^T = A + B$$

And therefore $A + B$ is symmetric. Since A and B were arbitrary in W, it follows that W is closed under addition

Closed under scalar multiplication: Suppose A in in W and c is a real number, then $A^T = 0$, and so, by properties of transposes:

$$(cA)^T = c\left(A^T\right) = cA$$

And therefore cA is symmetric. Since A and c were arbitrary, it follows that W is closed under multiplication.

Therefore, W is a subspace of V, and hence a vector space.

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2. *(2 points)* Is the following set W a subspace of $\mathbb{R}^2$? Justify **briefly**.

$$W = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 0\right\}$$

Notice that $x^2 + y^2 = 0$ implies $x = 0$ and $y = 0$, so W is nothing other than $\{(0,0)\}$, the zero vector-space, which is a subspace of $\mathbb{R}^2$. Therefore, W is a subspace of $\mathbb{R}^2$.

3. *(4 points)* Find an explicit description of $Nul(A)$ by writing it as the span of some vectors, where A is the following matrix:

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the right-hand-side is not in reduced row-echelon form, let's further row-reduce it:

$$\begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 4 & 5 & -6 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(I first subtracted the second row from the first, and then subtracted 3 times the third row from the second and 4 times the third row from the first)

Now if $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \\ r \end{bmatrix}$, then we get:

$$\begin{cases} x - 2z + 9s + 2r = 0 \\ y - z + 7s + 3r = 0 \\ t - s - 2r = 0 \end{cases}$$

That is:

$$\begin{cases} x = 2z - 9s - 2r \\ y = z - 7s - 3r \\ t = s + 2r \end{cases}$$

Hence we get:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} 2z - 9s - 2r \\ z - 7s - 3r \\ z \\ s + 2r \\ s \\ r \end{bmatrix} = z \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

And therefore:

$$Nul(A) = Span \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(in fact, you'll see later that this set is also a basis for $Nul(A)$)