MATH 54 – QUIZ 6 – SOLUTIONS

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1. *(6 points)*
   (a) *(4 points)* Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & -3 & 1 \\ -3 & -3 & 1 \end{bmatrix}$$

Eigenvalues:

$$\begin{aligned}
\det(\lambda I - A) =& \det \begin{bmatrix} \lambda - 1 & -1 & 1 \\ 3 & \lambda + 3 & -1 \\ 3 & 3 & \lambda - 1 \end{bmatrix} \\
=&(\lambda - 1)((\lambda + 3)(\lambda - 1) + 3) + (3(\lambda - 1) + 3) + (9 - 3(\lambda + 3)) \\
=&(\lambda - 1)((\lambda + 3)(\lambda - 1) + 3) + \cancel{3\lambda} - \cancel{3} + \cancel{3} + \cancel{9} - \cancel{3\lambda} - \cancel{9} \\
=&(\lambda - 1)((\lambda + 3)(\lambda - 1) + 3) \\
=&(\lambda - 1)(\lambda^2 + 2\lambda - 3 + 3) \\
=&(\lambda - 1)(\lambda^2 + 2\lambda) \\
=&(\lambda - 1)(\lambda)(\lambda + 2) \\
=&0
\end{aligned}$$

which gives us $\boxed{\lambda = -2, 0, 1}$

Eigenvectors:

$\lambda = -2$

$$Nul(2I - A) = Nul \begin{bmatrix} -3 & -1 & 1 \\ 3 & 1 & -1 \\ 3 & 3 & -3 \end{bmatrix} \to Nul \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = Span \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

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$\underline{\lambda = 0}$ (I gave you that one)

$$Nul(A) = Span\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$$

$\underline{\lambda = 1}$ (I gave you that one)

$$Nul(I - A) = Span\left\{\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$$

Answer:

$$D = \begin{bmatrix}-2 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}, \quad P = \begin{bmatrix}0 & -1 & -1\\1 & 1 & 1\\1 & 0 & 1\end{bmatrix}$$

(b) *(2 points)* Suppose $\mathbf{v} = \mathbf{v_1} + \mathbf{v_2} - \mathbf{v_3}$, where $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are the three eigenvectors you found above. Find $A^{2015}\mathbf{v}$

$$\begin{aligned}A^{2015}\mathbf{v} =& A^{2015}\left(\mathbf{v_1} + \mathbf{v_2} - \mathbf{v_3}\right)\\ =& A^{2015}\mathbf{v_1} + A^{2015}\mathbf{v_2} - A^{2015}\mathbf{v_3}\\ =& (-2)^{2015}\begin{bmatrix}0\\1\\1\end{bmatrix} + (0)^{2015}\begin{bmatrix}-1\\1\\0\end{bmatrix} - 1^{2015}\begin{bmatrix}-1\\1\\1\end{bmatrix}\\ =& \begin{bmatrix}1\\(-2)^{2015} - 1\\(-2)^{2015} - 1\end{bmatrix}\\ =& \begin{bmatrix}1\\-\left(2^{2015}\right) - 1\\-\left(2^{2015}\right) - 1\end{bmatrix}\end{aligned}$$

2. *(4 points)* Define $T : M_{2\times 2} \to M_{2\times 2}$ by:

$$T(A) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} A$$

Find the matrix of T relative to the basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ of } M_{2\times 2}$$

$$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

Hence the matrix of T is:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Note: It is incorrect to write $\begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$. It's **very** important to convert the vectors you found into *column* vectors!!!