MATH 54 – TRUE/FALSE QUESTIONS FOR MIDTERM 1 – SOLUTIONS

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1. (a) **TRUE** If the **augmented** matrix of the system $A\mathbf{x} = \mathbf{b}$ has a pivot in the last column, then the system $A\mathbf{x} = \mathbf{b}$ has no solution.

   (that's because there's a row of the form $\begin{bmatrix} 0 & 0 & \cdots & 0 \; b \end{bmatrix}$, where $b \neq 0$)

   (b) **FALSE** If A and B are invertible 2×2 matrices, then $(AB)^{-1} = A^{-1}B^{-1}$

   (it's $(AB)^{-1} = B^{-1}A^{-1}$, *reverse order*)

   (c) **TRUE** If A is a 3×3 matrix such that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^3$.

   (the IMT implies that A is invertible, and the IMT again implies the desired result)

   (d) **TRUE** The general solution to $A\mathbf{x} = \mathbf{b}$ is of the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_0$, where $\mathbf{x}_p$ is a *particular* solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_0$ is the *general* solution to $A\mathbf{x} = \mathbf{0}$.

   (See section 1.5)

   (e) **TRUE** If P and D are $n \times n$ matrices, then $det(PDP^{-1}) = det(D)$

$$det(PDP^{-1}) = det(P)det(D)det(P^{-1}) = \cancel{det(P)}det(D)\frac{1}{\cancel{det(P)}} = det(D)$$

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(f) **FALSE** If $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, then $Nul(T) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$$Nul(T) = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \text{ s.t. } \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \text{ s.t.} x = 0\right\} = \left\{\begin{bmatrix} 0 \\ y \end{bmatrix}\right\} = Span\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

(g) **TRUE** The set of polynomials $\mathbf{p}$ in P_2 such that $\mathbf{p}(3) = 0$ is a subspace of P_2

(You can easily check that the 0-polynomial is in it, that it is closed under addition and scalar multiplication)

(h) **FALSE** $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$ (it's not even a *subset* of $\mathbb{R}^3$!!!)

(a) **FALSE** If A and B are any 2×2 matrices, then $AB = BA$

Take for example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then:

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad BA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

which are not equal to each other!

(in fact, almost any two matrices you chose will give you a counterexample! The most important thing is that you had to find explicit A and B and you had to show that $AB \neq BA$)

(b) **TRUE** The matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ is not invertible.

Notice that the first and the third column of the matrix are equal, hence the columns of A are linearly dependent, so by the IMT A is not invertible!

Note: Many many other answers were possible! For example, you could calculate $det(A) = 0$, or you could row-reduce and say that the matrix has only 2 pivots. Any of those answers is acceptable!

(c) **TRUE** The set of matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is a subspace of $M_{2\times 2}$.
If you denote that set by V, then you get:

$$V = Span\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

And since the span of anything is a vector space, V is a vector space, and hence a subspace of $M_{2\times 2}$.

Alternatively you could have shown in the usual way that the O matrix is in it, and that it is closed under addition and scalar multiplication.

(d) **TRUE** The matrix of the linear transformation T which reflects points about the x-axis and then about the y-axis is the same as the matrix of the linear transformation S which rotates points about the origin by 180 degrees counterclockwise.

Calculate $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
Hence the matrix of T is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Calculate $S\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $S\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$
Hence the matrix of S is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
And notice the two matrices are the same!

2. (e) **TRUE** The following set is a basis for P_2: $\{1, 1+t, 1+t+t^2\}$.

Linear independence: Suppose $a(1)+b(1+t)+c(1+t+t^2) = 0$, then $(a+b+c)+(b+c)t+ct^2 = 0$, hence $c = 0$, hence $b = 0$, hence $a = 0$, hence $a = b = c = 0$, and the polynomials are linearly independent.

Span: Since P_2 is 3-dimensional, and the set contains 3 elements, hence the set also spans P_2

Therefore the set is a basis for P_2.

Note: There were many, many, many other ways to show why this is true! One way is to consider the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and notice its determinant is $1 \neq 0$, hence it is invertible, hence its columns are linearly independent and span $\mathbb{R}^3$.

(f) **FALSE** If V is a set that contains the $\mathbf{0}$-vector, and such that whenever $\mathbf{u}$ and $\mathbf{v}$ are in V, then $\mathbf{u}+\mathbf{v}$ is in V, then V is a vector space!

Consider the set $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, y \geq 0 \right\}$ in $\mathbb{R}^2$. (i.e. the upper-half-plane)

$\mathbf{0}$-vector: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in it!

Closed under addition: Suppose $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ are in V, then $y \geq 0$ and $y' \geq 0$. Then $\mathbf{u}+\mathbf{v} = \begin{bmatrix} x+x' \\ y+y' \end{bmatrix}$. But since $y + y' \geq 0$, we get $\mathbf{u}+\mathbf{v}$ is in V

Not closed under scalar multiplication: For example, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in V, but $(-2)\mathbf{u} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ is not in V.

3. (a) If A and B are square matrices, then $(A+B)^{-1} = A^{-1} + B^{-1}$.

FALSE

For example, take $A = [2]$ and $B = [3]$. Then the statement says: Is $\frac{1}{2+3} = \frac{1}{2} + \frac{1}{3}$? Which is not true.

Other explanation: Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A + B$ is the zero matrix, whose inverse is not defined, while the right-hand-side gives you 0.

(b) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one linear transformation, then T is also onto.

TRUE

Let A be the matrix of T. Then, if T is one-to-one, then A is invertible (by one of the conditions of invertibility), and hence, by another condition of invertibility, this implies that T is onto. Note that is works precisely because $m = n$, the result doesn't hold in general!

(c) If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly independent vectors in $\mathbb{R}^n$, then $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent as well!

TRUE

Suppose $a\mathbf{v_1} + b\mathbf{v_2} = \mathbf{0}$.

Goal: We want to show $a = b = 0$.

Now here's a clever trick: Add $0\mathbf{v_3} = \mathbf{0}$ to both sides of the equation.

Then we get: $a\mathbf{v_1} + b\mathbf{v_2} + 0\mathbf{v_3} = \mathbf{0}$

In particular, if we let $c = 0$, then we get: $a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = \mathbf{0}$

But $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent, so $a = b = c = 0$.

In particular $a = b = 0$, which we wanted to show!

Note: I have to admit, this is a tricky proof! But it illustrates why it's important to write down what you want to show and what you know!

(d) If A is an invertible square matrix, then $(A^T)^{-1} = (A^{-1})^T$

TRUE

Let $B = (A^{-1})^T$. All we need to show is that $A^T B = BA^T = I$, because then $B = \left(A^T\right)^{-1}$, which is what we want to show.

But:

$$A^T B = A^T \left(A^{-1}\right)^T = \left(A^{-1}A\right)^T = I^T = I$$

Where in the first step, we used the property of transposes $(CD)^T = D^T C^T$.
Similarly:

$$BA^T = \left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T = I^T = I$$

Hence $A^T B = BA^T = I$, which is what we needed to show!

(e) If A is a 3×3 matrix with two pivot positions, then the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

TRUE

If A has two pivot positions, then it has a row of zeros, and hence, because A is a 3×3 matrix, the solution $A\mathbf{x} = \mathbf{0}$ has at least one free variable, hence the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution!

(f) If A and B are square matrices, then $det(A + B) = det(A) + det(B)$.

FALSE

For example, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Then $det(A) = 1$, $det(B) = 1$, but $det(A + B) = det(O) = 0$ (where O is the zero-matrix).

(g) If $Nul(A) = \{\mathbf{0}\}$, then A is invertible.

FALSE

Don't worry, this got me too! This statement *is* true if A is **SQUARE** ! But if A is not square, this statement is never true!

For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $Nul(A) = \{\mathbf{0}\}$, but A is not invertible, because it is not square.

(h) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$

FALSE!

$\mathbb{R}^2$ is not even a *subset* of $\mathbb{R}^3$!!! Don't confuse this with $\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$, which *is* a subspace of $\mathbb{R}^3$ and very similar to $\mathbb{R}^2$ (but not exactly the same)

(i) If W is a subspace of V and $\mathcal{B}$ is a basis for V, then some subset of $\mathcal{B}$ is a basis for W.

FALSE

This is also very tricky (this got me too :)), because the 'opposite' statement does hold, namely if $\mathcal{B}$ is a basis for W, you can always complete $\mathcal{B}$ to become a basis of V (this is the 'basis extension theorem').

As a counterexample, take $V = \mathbb{R}^3$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and $W = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (a line in $\mathbb{R}^3$).

If the statement was true, then one of the vectors in $\mathcal{B}$ would be a basis for W, but this is bogus.

4. (a) **FALSE**

   (b) **FALSE**

   (c) **TRUE**

   (d) **TRUE**

   (e) **TRUE**

   (f) **FALSE** (Take $V = \mathbb{R}^2$ with the standard basis, and $W = Span \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. You cannot delete vectors from the standard basis to form a basis for W)

   (g) **TRUE**

   (h) **FALSE**

   (i) **TRUE**

   (j) **FALSE**