MATH 54 – TRUE/FALSE QUESTIONS FOR MIDTERM 2 – SOLUTIONS

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1. (a) [TRUE] If \( A \) is diagonalizable, then \( A^3 \) is diagonalizable.

\[ A = PDP^{-1}, \ \text{so} \ \ A^3 = P D^3 P = \widetilde{P} \widetilde{D} \widetilde{P}^{-1}, \ \text{where} \ \widetilde{P} = P \ \text{and} \ \widetilde{D} = D^3, \ \text{which is diagonal} \]

(b) [TRUE] If \( A \) is a \( 3 \times 3 \) matrix with 3 (linearly independent) eigenvectors, then \( A \) is diagonalizable

(This is one of the facts we talked about in lecture, the point is that to figure out if \( A \) is diagonalizable, look at the eigenvectors)

(c) [TRUE] If \( A \) is a \( 3 \times 3 \) matrix with eigenvalues \( \lambda = 1, 2, 3 \), then \( A \) is invertible

(No eigenvalue which is 0, so by the IMT, \( A \) is invertible)

(d) [TRUE] If \( A \) is a \( 3 \times 3 \) matrix with eigenvalues \( \lambda = 1, 2, 3 \), then \( A \) is (always) diagonalizable

(this is the useful test we’ve been talking about in lecture, \( A \) is diagonalizable since it has 3 distinct eigenvalues)

(e) [FALSE] If \( A \) is a \( 3 \times 3 \) matrix with eigenvalues \( \lambda = 1, 2, 2 \), then \( A \) is (always) not diagonalizable

(Take \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \), it is diagonal, hence diagonalizable)

Date: Monday, April 13th, 2015.
(f) **FALSE** If $\tilde{x}$ is the orthogonal projection of $x$ on $W$, then $\tilde{x}$ is orthogonal to $x$.

(Draw a picture)

(g) **FALSE** If $\hat{u}$ is the orthogonal projection of $u$ on $Span \{v\}$, then:

$$\hat{u} = \left( \frac{u \cdot v}{v \cdot v} \right) u$$

(It’s $\hat{u} = \left( \frac{u \cdot v}{v \cdot v} \right) v$, it has to be a multiple of $v$)

(h) **TRUE** If $Q$ is an orthogonal matrix, then $Q$ is invertible.

(Remember that in this course, orthogonal matrices are square)

2. (a) **FALSE** If $A$ is diagonalizable, then it is invertible.

For example, take $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. It is diagonalizable because it is diagonal, but it is not invertible!

(b) **FALSE** If $A$ is invertible, then $A$ is diagonalizable

Take $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (this is the ‘magic counterexample’ we talked about in lecture). It is invertible because $det(A) = 1 \neq 0$. To show it is not diagonalizable, let’s find the eigenvalues and eigenvectors of $A$:

**Eigenvalues:**

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$

Which gives us $\lambda = 1$.

**Eigenvectors:**

$$Nul(I - A) = Nul \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$
Which gives \(-y = 0\), so \(y = 0\), hence:

\[
Nul(I - A) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}
\]

Since there is only one (linearly independent) eigenvector, \(A\) is not diagonalizable!

3.  
1. (30 points, 5 pts each)

Label the following statements as T or F.  

Make sure to JUSTIFY YOUR ANSWERS!!! You may use any facts from the book or from lecture.

(a) If \(A = \{a_1, a_2, a_3\}\) and \(D = \{d_1, d_2, d_3\}\) are bases for \(V\), and \(P\) is the matrix whose \(i\)th column is \([d_i]_A\), then for all \(x \in V\), we have \([x]_D = P[x]_A\)

\[\text{FALSE}\]

First of all, \(P = \begin{bmatrix} [d_1]_A & [d_2]_A & [d_3]_A \end{bmatrix} = A \leftarrow D\)  

(remember, you always evaluate with respect to the new, cool basis, here it is \(A\)), so we should have:

\([x]_A = A \leftarrow D [x]_D = P[x]_D\)

And not the opposite!

(b) A 3 \times 3 matrix \(A\) with only one eigenvalue cannot be diagonalizable

\[\text{SUPER FALSE!!!!!!!!}\]
Remember that to check if a matrix is not diagonalizable, you really have to look at the eigenvectors!

For example, \( A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \) has only eigenvalue 2, but is diagonalizable (it’s diagonal!). Or you can choose \( A \) to be the \( O \) matrix, or the identity matrix, this also works!

(c) If \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are 2 eigenvectors of \( A \) corresponding to 2 different eigenvalues \( \lambda_1 \) and \( \lambda_2 \), then \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly independent!

TRUE (finally!)

Note: The proof is a bit complicated, but I’ve seen this on a past exam! I think at that point, the professor wanted to get revenge on his students for not coming to lecture!

Remember that eigenvectors have to be nonzero!

Now, assume \( a\mathbf{v}_1 + b\mathbf{v}_2 = 0 \).

Then apply \( A \) to this to get:

\[ A(a\mathbf{v}_1 + b\mathbf{v}_2) = A(0) = 0 \]

That is:

\[ aA(\mathbf{v}_1) + bA(\mathbf{v}_2) = 0 \]

\[ a\lambda_1\mathbf{v}_1 + b\lambda_2\mathbf{v}_2 = 0 \]

However, we can also multiply the original equation by \( \lambda_1 \) to get:

\[ a\lambda_1\mathbf{v}_1 + b\lambda_1\mathbf{v}_2 = 0 \]

Subtracting this equation from the one preceding it, we get:

\[ b(\lambda_1 - \lambda_2)\mathbf{v}_2 = 0 \]

So
\[ b(\lambda_1 - \lambda_2) = 0 \]

But \( \lambda_1 \neq \lambda_2 \), so \( \lambda_1 - \lambda_2 \neq 0 \), hence we get \( b = 0 \).

But going back to the first equation, we get:

\[ \begin{align*}
    a v_1 &= 0 \\
    \text{So } a &= 0. \\
    \text{Hence } a = b &= 0, \text{ and we’re done!}
\end{align*} \]

(d) If a matrix \( A \) has orthogonal columns, then it is an orthogonal matrix.

\textbf{FALSE}

Remember that an \textbf{orthogonal} matrix has to have \textbf{orthonormal} columns!

(e) For every subspace \( W \) and every vector \( y \), \( y - \text{Proj}_W y \) is orthogonal to \( \text{Proj}_W y \) (proof by picture is ok here)

\textbf{TRUE}

Draw a picture! \( \text{Proj}_W y \) is just another name for \( \hat{y} \).

(f) If \( y \) is already in \( W \), then \( \text{Proj}_W y = y \)

\textbf{TRUE}

Again, draw a picture!

If you want a more mathematical proof, here it is:

Let \( B = \{ w_1, \cdots w_p \} \) be an orthogonal basis for \( W \) (\( p = \text{Dim}(W) \)).
Then \( y = \left( \frac{y \cdot w_1}{w_1 \cdot w_1} \right) w_1 + \cdots + \left( \frac{y \cdot w_p}{w_p \cdot w_p} \right) w_p. \)

But then, by definition of \( \text{Proj}_W y = \hat{y}, \) we get:

\[
\hat{y} = \left( \frac{y \cdot w_1}{w_1 \cdot w_1} \right) w_1 + \cdots + \left( \frac{y \cdot w_p}{w_p \cdot w_p} \right) w_p = y
\]

So \( \hat{y} = y \) in this case.

4. (a) If \( A \) is a \( 3 \times 3 \) matrix with eigenvalues \( \lambda = 0, 2, 3 \), then \( A \) must be diagonalizable!

**TRUE** (an \( n \times n \) matrix with 3 distinct eigenvalues is diagonalizable)

(b) There does not exist a \( 3 \times 3 \) matrix \( A \) with eigenvalues \( \lambda = 1, -1, -1 + i \).

**TRUE** (here we assume \( A \) has real entries; eigenvalues always come in complex conjugate pairs, i.e. if \( A \) has eigenvalue \( -1 + i \), it must also have eigenvalue \( -1 - i \))

(c) If \( A \) is a symmetric matrix, then all its eigenvectors are orthogonal.

**FALSE**: Take \( A \) to be your favorite symmetric matrix, and, for example, take \( v \) to be one eigenvector, and \( w \) to be the *same* eigenvector (or a different eigenvector corresponding to
the same eigenvalue). That’s why we had to apply the Gram Schmidt process to each eigenspace in the previous problem!

(d) If $Q$ is an orthogonal $n \times n$ matrix, then $Row(Q) = Col(Q)$.

**TRUE**: (since $Q$ is orthogonal, $Q^T Q = I$, so $Q$ is invertible, hence $Row(Q) = Col(Q) = \mathbb{R}^n$)

(e) The equation $Ax = b$, where $A$ is a $n \times n$ matrix always has a unique least-squares solution.

**FALSE**: Take $A$ to be the zero matrix, and $b$ to be the zero vector! This statement is true if $A$ has rank $n$.

(f) If $AB = I$, then $BA = I$.

**FALSE**: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $AB = I$, but $BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(g) If $A$ is a square matrix, then $Rank(A) = Rank(A^2)$

**FALSE**: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $Rank(A) = 1$, but $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $Rank(A^2) = 0$. 
(h) If $W$ is a subspace, and $Py$ is the orthogonal projection of $y$ onto $W$, then $P^2y = Py$

**TRUE** (draw a picture! If you orthogonally project $Py = \hat{y}$ on $W$, you get $\hat{y}$)

(i) If $T : V \to W$, where $\text{dim}(V) = 3$ and $\text{dim}(W) = 2$, then $T$ cannot be one-to-one.

**TRUE** (by Rank-Nullity theorem, $\text{dim}(\text{Nul}(T)) + \text{Rank}(T) = 3$. But $\text{Rank}(T)$ can only be at most $\text{dim}(W) = 2$, so $\text{dim}(\text{Nul}(T)) > 0$, so $\text{Nul}(T) \neq \{0\}$)

(j) If $A$ is similar to $B$, then $\det(A) = \det(B)$.

**TRUE** (If $A = PBP^{-1}$, then $\det(A) = \det(B)$)