MATH 54 – TRUE/FALSE QUESTIONS FOR MIDTERM 2 – SOLUTIONS

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1. (a) **TRUE** If A is diagonalizable, then A^3 is diagonalizable.

   ($A = PDP^{-1}$, so $A^3 = PD^3P = \widetilde{P}\widetilde{D}\widetilde{P}^{-1}$, where $\widetilde{P} = P$ and $\widetilde{D} = D^3$, which is diagonal)

   (b) **TRUE** If A is a 3×3 matrix with 3 (linearly independent) eigenvectors, then A is diagonalizable

   (This is one of the facts we talked about in lecture, the point is that to figure out if A is diagonalizable, look at the eigenvectors)

   (c) **TRUE** If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 3$, then A is invertible

   (No eigenvalue which is 0, so by the IMT, A is invertible)

   (d) **TRUE** If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 3$, then A is (always) diagonalizable

   (this is the useful test we've been talking about in lecture, A is diagonalizable since it has 3 distinct eigenvalues)

   (e) **FALSE** If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 2$, then A is (always) not diagonalizable

   (Take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, it is diagonal, hence diagonalizable)

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(f) **FALSE** If $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{x}$ on W, then $\hat{\mathbf{x}}$ is orthogonal to $\mathbf{x}$.
(Draw a picture)

(g) **FALSE** If $\hat{\mathbf{u}}$ is the orthogonal projection of $\mathbf{u}$ on $Span\{\mathbf{v}\}$, then:

$$\hat{\mathbf{u}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{u}$$

(It's $\hat{\mathbf{u}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$, it has to be a multiple of $\mathbf{v}$)

(h) **TRUE** If Q is an orthogonal matrix, then Q is invertible.
(Remember that in this course, orthogonal matrices are square)

2. (a) **FALSE** If A is diagonalizable, then it is invertible.

For example, take $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. It is diagonalizable **because it is diagonal**, but it is not invertible!

(b) **FALSE** If A is invertible, then A is diagonalizable

Take $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (this is the 'magic counterexample' we talked about in lecture). It is invertible because $det(A) = 1 \neq 0$. To show it is not diagonalizable, let's find the eigenvalues and eigenvectors of A:

<u>Eigenvalues:</u>

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$

Which gives us $\lambda = 1$.

<u>Eigenvectors:</u>

$$Nul(I - A) = Nul \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Which gives $-y = 0$, so $y = 0$, hence:

$$Nul(I - A) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Since there is only one (linearly independent) eigenvector, A is not diagonalizable!

3.
1. *(30 points, 5 pts each)*

Label the following statements as **T** or **F**.

Make sure to **JUSTIFY YOUR ANSWERS!!!** You may use any facts from the book or from lecture.

(a) If $\mathcal{A} = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ and $\mathcal{D} = \{\mathbf{d_1}, \mathbf{d_2}, \mathbf{d_3}\}$ are bases for V, and P is the matrix whose ith column is $[\mathbf{d_i}]_{\mathcal{A}}$, then for all $\mathbf{x}$ in V, we have $[\mathbf{x}]_{\mathcal{D}} = P\,[\mathbf{x}]_{\mathcal{A}}$

FALSE

First of all, $P = \left[[\mathbf{d_1}]_{\mathcal{A}} \quad [\mathbf{d_2}]_{\mathcal{A}} \quad [\mathbf{d_3}]_{\mathcal{A}} \right] = \mathcal{A} \overset{P}{\leftarrow} \mathcal{D}$
(remember, you always evaluate with respect to the new, cool basis, here it is $\mathcal{A}$), so we should have:

$$[\mathbf{x}]_{\mathcal{A}} = \mathcal{A} \overset{P}{\leftarrow} \mathcal{D}\,[\mathbf{x}]_{\mathcal{D}} = P\,[\mathbf{x}]_{\mathcal{D}}$$

And not the opposite!

(b) A 3×3 matrix A with only one eigenvalue cannot be diagonalizable

SUPER FALSE!!!!!!!!!!

Remember that to check if a matrix is not diagonalizable, you really have to look at the eigenvectors!

For example, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ has only eigenvalue 2, but is diagonalizable (it's diagonal!). Or you can choose A to be the O matrix, or the identity matrix, this also works!

(c) If $\mathbf{v_1}$ and $\mathbf{v_2}$ are 2 eigenvectors of A corresponding to 2 **different** eigenvalues λ_1 and λ_2, then $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent!

TRUE (finally!)

Note: The proof is a bit complicated, but I've seen this on a past exam! I think at that point, the professor wanted to get revenge on his students for not coming to lecture!

Remember that eigenvectors have to be nonzero!

Now, assume $a\mathbf{v_1} + b\mathbf{v_2} = \mathbf{0}$.

Then apply A to this to get:

$$A(a\mathbf{v_1} + b\mathbf{v_2}) = A(\mathbf{0}) = \mathbf{0}$$

That is:

$$aA(\mathbf{v_1}) + bA(\mathbf{v_2}) = \mathbf{0}$$

$$a\lambda_1\mathbf{v_1} + b\lambda_2\mathbf{v_2} = \mathbf{0}$$

However, we can also multiply the original equation by λ_1 to get:

$$a\lambda_1\mathbf{v_1} + b\lambda_1\mathbf{v_2} = \mathbf{0}$$

Subtracting this equation from the one preceding it, we get:

$$b(\lambda_1 - \lambda_2)\mathbf{v_2} = \mathbf{0}$$

So

$$b(\lambda_1 - \lambda_2) = \mathbf{0}$$

But $\lambda_1 \neq \lambda_2$, so $\lambda_1 - \lambda_2 \neq 0$, hence we get $b = 0$.

But going back to the first equation, we get:

$$a\mathbf{v_1} = \mathbf{0}$$

So $a = 0$.

Hence $a = b = 0$, and we're done!

(d) If a matrix A has orthogonal columns, then it is an orthogonal matrix.

FALSE

Remember that an ortho**gonal** matrix has to have ortho**normal** columns!

(e) For every subspace W and every vector $\mathbf{y}$, $\mathbf{y} - Proj_W\mathbf{y}$ is orthogonal to $Proj_W\mathbf{y}$ (proof by picture is ok here)
TRUE

Draw a picture! $Proj_W\mathbf{y}$ is just another name for $\hat{y}$.

(f) If $\mathbf{y}$ is already in W, then $Proj_W\mathbf{y} = \mathbf{y}$

TRUE

Again, draw a picture!

If you want a more mathematical proof, here it is:

Let $\mathcal{B} = \{\mathbf{w_1}, \cdots \mathbf{w_p}\}$ be an orthogonal basis for W ($p = Dim(W)$).

Then $y = \left(\frac{\mathbf{y}\cdot\mathbf{w_1}}{\mathbf{w_1}\cdot\mathbf{w_1}}\right)\mathbf{w_1} + \cdots + \left(\frac{\mathbf{y}\cdot\mathbf{w_p}}{\mathbf{w_p}\cdot\mathbf{w_p}}\right)\mathbf{w_p}$.

But then, by definition of $Proj_W\mathbf{y} = \hat{\mathbf{y}}$, we get:

$$\hat{y} = \left(\frac{\mathbf{y}\cdot\mathbf{w_1}}{\mathbf{w_1}\cdot\mathbf{w_1}}\right)\mathbf{w_1} + \cdots + \left(\frac{\mathbf{y}\cdot\mathbf{w_p}}{\mathbf{w_p}\cdot\mathbf{w_p}}\right)\mathbf{w_p} = y$$

So $\hat{\mathbf{y}} = \mathbf{y}$ in this case.

4. (a) If A is a 3×3 matrix with eigenvalues $\lambda = 0, 2, 3$, then A must be diagonalizable!

   TRUE (an $n \times n$ matrix with 3 distinct eigenvalues is diagonalizable)

   (b) There does not exist a 3×3 matrix A with eigenvalues $\lambda = 1, -1, -1+i$.

   TRUE (here we assume A has real entries; eigenvalues always come in complex conjugate pairs, i.e. if A has eigenvalue $-1+i$, it must also have eigenvalue $-1-i$)

   (c) If A is a symmetric matrix, then all its eigenvectors are orthogonal.

   FALSE: Take A to be your favorite symmetric matrix, and, for example, take $\mathbf{v}$ to be one eigenvector, and $\mathbf{w}$ to be the *same* eigenvector (or a different eigenvector corresponding to the same eigenvalue). That's why we had to apply the Gram Schmidt process to each eigenspace in the previous problem!

(d) If Q is an orthogonal $n \times n$ matrix, then $Row(Q) = Col(Q)$.

TRUE: (since Q is orthogonal, $Q^T Q = I$, so Q is invertible, hence $Row(Q) = Col(Q) = \mathbb{R}^n$)

(e) The equation $A\mathbf{x} = \mathbf{b}$, where A is a $n \times n$ matrix always has a unique least-squares solution.

FALSE: Take A to be the zero matrix, and $\mathbf{b}$ to be the zero vector! This statement is true if A has rank n.

(f) If $AB = I$, then $BA = I$.

FALSE: Let $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $AB = I$, but $BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$!

(g) If A is a square matrix, then $Rank(A) = Rank(A^2)$

FALSE: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $Rank(A) = 1$, but $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $Rank(A^2) = 0$.

(h) If W is a subspace, and $P\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto W, then $P^2\mathbf{y} = P\mathbf{y}$

TRUE (draw a picture! If you orthogonally project $P\mathbf{y} = \hat{\mathbf{y}}$ on W, you get $\hat{\mathbf{y}}$)

(i) If $T : V \to W$, where $dim(V) = 3$ and $dim(W) = 2$, then T cannot be one-to-one.

TRUE (by Rank-Nullity theorem, $dim(Nul(T))+Rank(T) = 3$. But $Rank(T)$ can only be at most $dim(W) = 2$, so $dim(Nul(T)) > 0$, so $Nul(T) \neq \{\mathbf{0}\}$)

(j) If A is similar to B, then $det(A) = det(B)$.

TRUE (If $A = PBP^{-1}$, then $det(A) = det(B)$)