SOLUTIONS

1. *(10 points, 1 point each)* Label each statement as **TRUE** or **FALSE**. In this question, you do **NOT** have to justify your answer. Each correct answer will get 1 point and each incorrect or illegible answer will get 0 points.

T (a) If A is row-equivalent to B, then $Nul(A) = Nul(B)$.

T (b) If A has orthonormal columns, then the least-squares solution of $A\mathbf{x} = \mathbf{b}$ is $\widehat{\mathbf{x}} = A^T\mathbf{b}$ $\left(\underbrace{A^TA}_{I} \hat{x} = A^T b \Rightarrow \hat{x} = A^T b \right)$

F (c) If A has orthogonal columns, then the orthogonal projection of $\mathbf{x}$ on $Col(A)$ is $\widehat{\mathbf{x}} = AA^T\mathbf{x}$ (COLUMNS NEED TO BE ORTHONORMAL)

T (d) If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2 $(A^2v = AAv = A(\lambda v) = \lambda Av = \lambda\lambda v = \lambda^2 v)$

F (e) If A is a 2×3 matrix, then the set of solutions of $A\mathbf{x} = \begin{bmatrix}1\\0\end{bmatrix}$ is a subspace of $\mathbb{R}^3$ $\left(\begin{bmatrix}0\\0\\0\end{bmatrix} \right.$ IS NOT IN IT$)$

F (f) If T is linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ that is one-to-one, then T is also onto $\mathbb{R}^m$ $A = \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$ PIVOT IN EVERY COL, BUT NOT EVERY ROW

T (g) The pivot columns of A form a basis for the column space of A

F (h) If T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$, then the matrix A of T is a 4×3 matrix (3x4 MATRIX)

T (i) If A is similar to B, then B is similar to A $(A = PBP^{-1} \Rightarrow B = P^{-1}AP = QAQ^{-1}, Q = P^{-1})$

F (j) If $A^2 = O$ (the zero-matrix), then $A = O$ $\left(A = \begin{bmatrix}0&1\\0&0\end{bmatrix}, A^2 = \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix} \right)$

2. *(10 points)* Find the eigenvalues of

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$$

Note: You do **NOT** need to find the eigenvectors of A

$$|\lambda I - A| = \begin{vmatrix} \lambda-4 & -2 & -3 \\ 1 & \lambda-1 & 3 \\ -2 & -4 & \lambda-9 \end{vmatrix}$$

$$= (\lambda-4)\begin{vmatrix} \lambda-1 & 3 \\ -4 & \lambda-9 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ -2 & \lambda-9 \end{vmatrix} - 3\begin{vmatrix} 1 & \lambda-1 \\ -2 & -4 \end{vmatrix}$$

$$= (\lambda-4)\left[(\lambda-1)(\lambda-9)+12\right] + 2(\lambda-9+6) - 3(-4+2(\lambda-1))$$

$$= (\lambda-4)(\lambda^2-10\lambda+9+12) + 2(\lambda-3) - 3(2\lambda-2-4)$$

$$= (\lambda-4)(\lambda^2-10\lambda+21) + 2(\lambda-3) - 3(2\lambda-6)$$

$$= (\lambda-4)(\lambda-3)(\lambda-7) + 2(\lambda-3) - 6(\lambda-3)$$

$$= (\lambda-3)\left[(\lambda-4)(\lambda-7)+2-6\right]$$

$$= (\lambda-3)\left[\lambda^2-11\lambda+28-4\right]$$

$$= (\lambda-3)(\lambda^2-11\lambda+24)$$

$$= (\lambda-3)(\lambda-3)(\lambda-8)$$

$$= (\lambda-3)^2(\lambda-8) = 0$$

$$\Rightarrow \boxed{\lambda = 3 \text{ AND } \lambda = 8}$$

3. *(10 points)* Find a diagonal matrix D and an invertible matrix P with $A = PDP^{-1}$

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

Note: You can assume, and you do **NOT** need to show, that A has eigenvalues $\lambda = 1$ and $\lambda = 5$

1) $\underline{\lambda = 1}$ $\quad \text{NUL}(1I - A) = \text{NUL}\begin{bmatrix} 1-2 & -2 & 1 \\ -1 & 1-3 & 1 \\ 1 & 2 & 1-2 \end{bmatrix}$

$$= \text{NUL}\begin{bmatrix} -1 & -2 & 1 \\ -1 & -2 & 1 \\ 1 & 2 & -1 \end{bmatrix} = \text{NUL}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x + 2y - z = 0 \Rightarrow x = -2y + z$$

$$\underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y+z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$E_1 = \text{SPAN}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

2) $\underline{\lambda = 5}$ $\quad \text{NUL}(5I - A) = \text{NUL}\begin{bmatrix} 5-2 & -2 & 1 \\ -1 & 5-3 & 1 \\ 1 & 2 & 5-2 \end{bmatrix}$

$$= \text{NUL}\begin{bmatrix} 3 & -2 & 1 \\ -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{NUL}\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 3 & -2 & 1 \end{bmatrix} (\times 1) \; (\times -3)$$

$$= \text{NUL}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & -8 & -8 \end{bmatrix} = \text{NUL}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} (\times -2) = \text{NUL}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x + z = 0 \\ y + z = 0 \end{cases} \Rightarrow \begin{cases} x = -z \\ y = -z \end{cases} \Rightarrow \underline{X} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad E_5 = \text{SPAN}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

3) $\underline{\text{ANS}}$ $\quad \boxed{D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}}, \quad \boxed{P = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}}$

4. *(10 points)* Use Cramer's rule to solve the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \text{[redacted]} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

1) $\text{DET}(A) = \begin{vmatrix} \boxed{3 \quad 5 \quad 4} \\ 1 \quad 2 \quad 1 \\ 2 \quad 1 \quad 1 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$

$= 3(2-1) - 5(1-2) + 4(1-4) = 3 + 5 - 12 = -4$

2) $x = \dfrac{\begin{vmatrix} 1 & 5 & 4 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix}}{-4} = \dfrac{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 5 & 4 \\ 2 & 1 \end{vmatrix}}{-4} = \dfrac{2-1-5+8}{-4}$

$= \dfrac{4}{-4} = -1$

$y = \dfrac{\begin{vmatrix} 3 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{vmatrix}}{-4} = \dfrac{-\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix}}{-4} = \dfrac{-1+2+3-4}{-4} = 0$

$z = \dfrac{\begin{vmatrix} 3 & 5 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & -1 \end{vmatrix}}{-4} = \dfrac{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix}}{-4} = \dfrac{1-4-6+5}{-4} = \dfrac{-4}{-4}$

$= 1$

3) $\underline{X} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

5. *(10 = 4 + 1 + 2 + 1 + 2 points)* In this question, there will be no partial credit for each sub-part.

For the following matrix A, find:

(a) A basis for $Nul(A)$
(b) $\dim(Nul(A))$
(c) A basis for $Col(A)$
(d) $Rank(A)$
(e) State the Rank Theorem

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 \\ 1 & 2 & -4 & 10 & 13 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -3 & 1 & -5 & -7 \\ 1 & -2 & 0 & 0 & -5 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 \\ 0 & 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (x-3) \quad (x-7)$$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 0 & 16 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (x-1)$$

$$\sim \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 0 & 9 \\ 0 & 1 & -1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x & y & z & t & s \end{matrix} \end{array} \quad \underline{\text{RREF}}$$

(a) $\begin{cases} x - 2z + 9s = 0 \\ y - z + 7s = 0 \\ t - s = 0 \end{cases}$

$\Rightarrow \begin{cases} x = 2z - 9s \\ y = z - 7s \\ t = s \end{cases}$

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \end{bmatrix} = \begin{bmatrix} 2z-9s \\ z-7s \\ z \\ s \\ s \end{bmatrix} = z\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

<u>BASIS FOR NUL(A)</u> $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\dim(Nul(A)) = 2$

(c) <u>BASIS FOR COL(A)</u> $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{bmatrix} \right\}$

(d) $RANK(A) = 3$

(e) $RANK(A) + \dim(NUL(A)) = N$

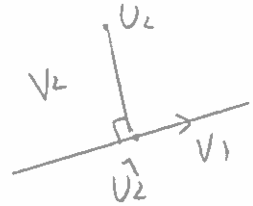
6. *(10 points)* Use the Gram-Schmidt Process to find an [redacted] basis for $H = Span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

ORTHOGONAL

$$\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

4) ANS $\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-2\\1\\1\end{bmatrix}\right\}$

1) $v_1 = u_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

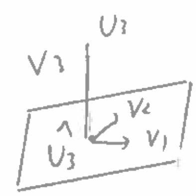


2) $\hat{u}_2 = \left(\dfrac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$

$= \left(\dfrac{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\right)\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \dfrac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}3/4\\3/4\\3/4\\3/4\end{bmatrix}$

$v_2 = u_2 - \hat{u}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \begin{bmatrix}3/4\\3/4\\3/4\\3/4\end{bmatrix} = \begin{bmatrix}-3/4\\1/4\\1/4\\1/4\end{bmatrix} \sim \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$

(CHECK: $v_2 \cdot v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = -3+1+1+1 = 0$ ✓)



3) $\hat{u}_3 = \left(\dfrac{u_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\dfrac{u_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$

$= \left(\dfrac{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\right)\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \left(\dfrac{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}}\right)\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$

$= \dfrac{2}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \dfrac{2}{12}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} + \begin{bmatrix}-1/2\\1/6\\1/6\\1/6\end{bmatrix} = \begin{bmatrix}0\\2/3\\2/3\\2/3\end{bmatrix}$

$v_3 = u_3 - \hat{u}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\2/3\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}0\\-2/3\\1/3\\1/3\end{bmatrix} \sim \begin{bmatrix}0\\-2\\1\\1\end{bmatrix}$

(CHECK $v_1 \cdot v_3 = -2+1+1 = 0$ ✓
$v_2 \cdot v_3 = -2+1+1 = 0$ ✓

7. *(10 points)* Using orthogonal projections (that is, by calculating $\widehat{\mathbf{b}}$) find the least-squares solution and the least-squares error of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{array}{c} \begin{array}{cc} U & V \end{array} \\ \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \end{array}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

1) $$\hat{b} = \left(\frac{b \cdot U}{U \cdot U}\right) U + \left(\frac{b \cdot V}{V \cdot V}\right) V$$

$$= \left(\frac{\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}\right) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}}\right) \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$= \left(\frac{9}{3}\right) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{12}{24} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$$

2) $$A\hat{x} = \hat{b} \Longrightarrow \begin{array}{c} (x1) \\ (x-1) \end{array} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ -1 & 4 & -1 \\ 1 & 2 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 6 & 3 \\ 0 & 0 & 0 \end{array}\right] (\div 6)$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right] (x-2) \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\hat{x} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$$

3) $$\text{Error} = \| b - \hat{b} \| = \left\| \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\| = \sqrt{2}$$

8. *(10 points, 5 points each)* Label each statement as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer, meaning that if the statement is true, you have to explain why it's true, and if the statement is false, you have to give an explicit counterexample and show why it's a counterexample

FALSE (a) For any 2×2 matrices A and B, $\det(A+B) = \det(A)+\det(B)$

LET $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

THEN $\text{DET}(A) + \text{DET}(B) = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 + 0 = 0$ $\neq$

BUT $\text{DET}(A+B) = \text{DET}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{DET}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$

FALSE (b) If A is invertible, then A is diagonalizable

LET $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

THEN $\text{DET}(A) = 1 \neq 0$, SO A IS INVERTIBLE

BUT A IS NOT DIAGONALIZABLE

WHY? $|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0 \Rightarrow \underline{\lambda = 1}$

$\underline{\lambda = 1}$ $\quad \text{NUL}(I - A) = \text{NUL}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad -y = 0 \Rightarrow y = 0$

$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \text{SPAN}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

A HAS ONLY 1 (LI) EIGENVECTOR, SO A IS NOT DIAGONALIZABLE.

9. *(10 points)* Let u be a (nonzero) eigenvector of A corresponding to λ and v be a (nonzero) eigenvector of A corresponding to μ, where $\lambda \neq \mu$. Show that u and v are linearly independent. Explain where you used the fact that $\lambda \neq \mu$

Hint: Start with the definition of linear independence of u and v. On the one hand, apply A to your equation. On the other hand, multiply your *original* equation by λ.

SUPPOSE $a\underline{u} + b\underline{v} = \underline{0}$

ON THE ONE HAND: $A(a\underline{u} + b\underline{v}) = A\underline{0}$

$$A(a\underline{u}) + A(b\underline{v}) = \underline{0}$$

$$aA\underline{u} + bA\underline{v} = \underline{0}$$

$$a\lambda\underline{u} + b\mu\underline{v} = \underline{0} \quad (1)$$

ON THE OTHER HAND: $a\underline{u} + b\underline{v} = \underline{0}$

$$\Rightarrow \lambda(a\underline{u} + b\underline{v}) = \lambda\underline{0}$$

$$\Rightarrow a\lambda\underline{u} + b\lambda\underline{v} = \underline{0} \quad (2)$$

$$(1) - (2) \Rightarrow (\cancel{a\lambda\underline{u}} + b\mu\underline{v}) - \cancel{a\lambda\underline{u}} - b\lambda\underline{v} = \underline{0} - \underline{0}$$

$$b\underbrace{(\mu - \lambda)}_{\neq 0 \text{ SINCE } \mu \neq \lambda}\underline{v} = \underline{0}$$

$$\Rightarrow b\underline{v} = \frac{1}{\mu - \lambda}\underline{0} = \underline{0}$$

$$\Rightarrow b\underline{v} = \underline{0} \Rightarrow \underline{b = 0} \quad (\text{SINCE } \underline{v} \neq \underline{0})$$

BUT THEN $a\underline{u} + b\underline{v} = \underline{0} \Rightarrow a\underline{u} + 0\underline{v} = \underline{0} \Rightarrow a\underline{u} = \underline{0} \Rightarrow \underline{a = 0}$ (SINCE $\underline{u} \neq \underline{0}$)

HENCE $a = 0$ AND $b = 0$ ✓

10. *(10 points)* The Grand Finale!!!

Welcome to the fierce final battle between Son Goku and Vegeta! Let s_n and v_n be the HP (Hit Points) of Son Goku and Vegeta after round n ~~and v_n be the HP of Vegeta~~. Assume that

$$\begin{cases} s_{n+1} = \frac{3}{2}s_n - \frac{1}{2}v_n \\ v_{n+1} = s_n + 0V_N \end{cases}$$

Assume that initially, ~~$s_0 = v_0 = 10$~~ (not over 9000, sadly)

$S_0 = 10,\ V_0 = 5$

This time, unlike the Pokemon battle, assume that it's ok to have negative HP (which just means that they continue battling in the afterlife).

Question: What happens to s_n and v_n as $n \to \infty$?

1)
$$\begin{bmatrix} S_{N+1} \\ V_{N+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 3/2 & -1/2 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} S_N \\ V_N \end{bmatrix}$$

2) DIAGONALIZE A

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3/2 & 1/2 \\ -1 & \lambda \end{vmatrix} = \lambda\left(\lambda - \tfrac{3}{2}\right) + \tfrac{1}{2}$$
$$= \lambda^2 - \tfrac{3}{2}\lambda + \tfrac{1}{2}$$
$$= (\lambda - 1)(\lambda - \tfrac{1}{2}) = 0 \Rightarrow \lambda = 1,\ \lambda = \tfrac{1}{2}$$

$\lambda = 1$ $\quad NUL(I - A) = NUL\begin{bmatrix} 1-3/2 & 1/2 \\ -1 & 1 \end{bmatrix} = NUL\begin{bmatrix} -1/2 & 1/2 \\ -1 & 1 \end{bmatrix} = NUL\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$\left(X - Y = 0 \Rightarrow X = Y \Rightarrow \underline{X} = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ X \end{bmatrix} = X\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \quad = SPAN\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\lambda = \frac{1}{2}$ $\quad NUL\left(\frac{1}{2}I - A\right) = NUL\begin{bmatrix} \frac{1}{2} - \frac{3}{2} & \frac{1}{2} \\ -1 & 1/2 \end{bmatrix} = NUL\begin{bmatrix} -1 & 1/2 \\ -1 & 1/2 \end{bmatrix} = NUL\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$

$\left(X - \frac{1}{2}Y = 0 \Rightarrow X = \frac{1}{2}Y \Rightarrow \underline{X} = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1/2\,Y \\ Y \end{bmatrix} = Y\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right) \quad = SPAN\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$

$= SPAN\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$A = PDP^{-1}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

3) Now $\begin{bmatrix} S_N \\ V_N \end{bmatrix} = A^N \begin{bmatrix} S_0 \\ V_0 \end{bmatrix} = A^N \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

so $\begin{bmatrix} S_\infty \\ V_\infty \end{bmatrix} = A^\infty \begin{bmatrix} 10 \\ 5 \end{bmatrix} = P D^\infty P^{-1} \begin{bmatrix} 10 \\ 15 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1^\infty & 0 \\ 0 & (1/2)^\infty \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 15 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 15 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ 15 \end{bmatrix}$$