MATH 3A – FINAL EXAM

Name: ________________________

Student ID: ________________________

Instructions: This is it, your final hurdle to freedom! Welcome to your Final Exam! You have 120 minutes to take this exam, for a total of 100 points. No books, notes, calculators, or cellphones are allowed. Remember that you are not only graded on your final answer, but also on your work. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. May your luck be diagonalizable! :)

Academic Honesty Statement: I hereby certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating (no matter how small) results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to section 102.1 of the UCI Student Code of Conduct.

Signature: __________________________________

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Date: Wednesday, March 20, 2019.

1. *(10 points, 1 point each)* Label each statement as **TRUE** or **FALSE**. In this question, you do **NOT** have to justify your answer. Each correct answer will get 1 point and each incorrect or illegible answer will get 0 points.

   (a) If A is row-equivalent to B, then $Nul(A) = Nul(B)$.

   (b) If A has orthonormal columns, then the least-squares solution of $A\mathbf{x} = \mathbf{b}$ is $\widehat{\mathbf{x}} = A^T\mathbf{b}$

   (c) If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2

   (d) If A is a 2×3 matrix, then the set of solutions $\mathbf{x}$ of $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a subspace of $\mathbb{R}^3$

   (e) If T is a one-to-one linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, then T must also be onto $\mathbb{R}^m$

   (f) The pivot columns of A always form a basis of $Col(A)$

   (g) If T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ and A is the matrix of T, then A is 4×3

   (h) If A has orthogonal columns, then the orthogonal projection of $\mathbf{x}$ on $Col(A)$ is $\widehat{\mathbf{x}} = AA^T\mathbf{x}$

   (i) If A is similar to B, then B must be similar to A

   (j) If A is 2×2 and $A^2 = O$ (the zero-matrix), then $A = O$

2. *(10 points)* Find the eigenvalues of

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$$

Note: Do **NOT** find the eigenvectors of A

3. *(10 points)* Find a diagonal matrix D and an invertible matrix P with $A = PDP^{-1}$

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

Note: You can assume, and do **NOT** need to show, that A has eigenvalues $\lambda = 1$ and $\lambda = 5$.

4. *(10 points)* Use Cramer's rule to solve the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

5. *(10 = 4 + 1 + 2 + 1 + 2 points)* In this question, there will be very little partial credit for each sub-part.

For the following matrix A, find:

(a) A basis for $Nul(A)$
(b) $\dim(Nul(A))$
(c) A basis for $Col(A)$
(d) $Rank(A)$
(e) State the Rank Theorem

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 \\ 1 & 2 & -4 & 10 & 13 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -3 & 1 & -5 & -7 \\ 1 & -2 & 0 & 0 & -5 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 \\ 0 & 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6. *(10 points)* Use the Gram-Schmidt Process to find an **orthogonal** basis for $H = Span\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$, where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

7. *(10 points)* Using **orthogonal projections** (that is, by calculating $\widehat{\mathbf{b}}$) find the least-squares solution **and** the least-squares error of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

8. *(10 points, 5 points each)* Label each statement as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer, meaning that if the statement is true, you have to explain why it's true, and if the statement is false, you have to give an explicit counterexample and **show why it's a counterexample**

   (a) For any 2×2 matrices A and B, $\det(A+B) = \det(A) + \det(B)$

   (b) If A is invertible, then A is diagonalizable

9. *(10 points)* Let $\mathbf{u}$ and $\mathbf{v}$ be (nonzero) eigenvectors of A corresponding respectively to λ and μ, where $\lambda \neq \mu$. Show that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Explain where you used the fact that $\lambda \neq \mu$

   Hint: Start with the definition of linear independence of $\mathbf{u}$ and $\mathbf{v}$. Then apply A to your equation. On the other hand, take your *original* equation and multiply it by λ.

10. *(10 points)* The Grand Finale!!!

Welcome to the fierce final battle between Son Goku and Vegeta! Let s_n and v_n be the HP (Hit Points) of Son Goku and Vegeta after round n. Assume that

$$\begin{cases} s_{n+1} = \dfrac{3}{2}\, s_n - \dfrac{1}{2}\, v_n \\ v_{n+1} = 1\, s_n + 0\, v_n \end{cases}$$

Given that $\begin{bmatrix} s_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find $\begin{bmatrix} s_\infty \\ v_\infty \end{bmatrix}$

Note: Unlike the Pokemon battle, here it's ok if s_n or v_n are negative (it just means both fight in the afterlife). Also, sadly the answer is not over 9000

(Scratch paper)