SOLUTIONS

1. *(10 points, 1 point each)* Label each statement as **TRUE** or **FALSE**. In this question, you do **NOT** have to justify your answer. Each correct answer will get 1 point and each incorrect or illegible answer will get 0 points.

T (a) If A is similar to I, then $A = I$ $\left(A \sim I \Rightarrow A = PIP^{-1} = PP^{-1} = I \right)$

T (b) If A is invertible and λ is an eigenvalue of A, then $\frac{1}{\lambda}$ must be an eigenvalue of A^{-1} $\left(AV = \lambda V \Rightarrow V = A^{-1}(\lambda V) = \lambda A^{-1}(V) = V \Rightarrow A^{-1}V = \frac{1}{\lambda} V \right)$

F (c) For any matrix A, $Col(A)$ is orthogonal to $Nul(A)$ $Col(A) \perp Nul(A^T)$

F (d) A 3×3 matrix with eigenvalues $\lambda = 0$ and $\lambda = 1$ can never be diagonalizable $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ DIAGONALIZABLE (IT'S DIAGONAL)

T (e) A 3×3 matrix with eigenvalues $\lambda = 0$ and $\lambda = 1$ can never be invertible (0 IS AN EIGENVALUE)

T (f) If A is a 2×3 matrix, then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is never one-to-one $\begin{bmatrix} * & & \\ & * & \end{bmatrix}$ ONLY 2 PIVOTS MAX $\Rightarrow$ 1 FREE VAR

F (g) If A is a 2×3 matrix, then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is never onto $\mathbb{R}^2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ PIVOT IN EVERY ROW $\Rightarrow$ ONTO

F (h) The least-squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ is a vector in $\mathbb{R}^n$ that satisfies $\|A\mathbf{x} - \mathbf{b}\| \leq \|A\hat{\mathbf{x}} - \mathbf{b}\|$ for all $\mathbf{x}$ $\|A\hat{x} - b\| \leq \|Ax - b\|$

T (i) If $\mathbf{u}$ is an eigenvector of A corresponding to $\lambda = 1$ and $\mathbf{v}$ is an eigenvector of A corresponding to $\lambda = -1$, then $A^{2019}(\mathbf{u} - \mathbf{v}) = \mathbf{u} + \mathbf{v}$ $A^{2019}(U - V) = A^{2019}U - A^{2019}V = 1^{2019}U - (-1)^{2019}V$

$= U - (-1)V$

$= U + V$

F (j) For any square matrix A, $\det(3A) = 3\det(A)$

$\left| 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 9$

$3 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 3$

2. *(10 points)* Find a diagonalizable matrix D and an invertible matrix P with $A = PDP^{-1}$, where

$$A = \begin{bmatrix} 4 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

3) $\boxed{D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$

1) $$|\lambda I - A| = \begin{vmatrix} \lambda-4 & 2 & 0 \\ 1 & \lambda-3 & 0 \\ 0 & 0 & \lambda-2 \end{vmatrix}$$

$$= (\lambda-2)\begin{vmatrix} \lambda-4 & 2 \\ 1 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-2)\left[(\lambda-4)(\lambda-3)-2\right]$$

$$= (\lambda-2)\left[\lambda^2 - 7\lambda + 12 - 2\right)$$

$$= (\lambda-2)(\lambda^2 - 7\lambda + 10)$$

$$= (\lambda-2)(\lambda-2)(\lambda-5)$$

$$= (\lambda-2)^2(\lambda-5) = 0 \implies \underline{\lambda = 2, 5}$$

2) $\underline{\lambda=2}$ $$\mathrm{NUL}(2I-A) = \mathrm{NUL}\begin{bmatrix} 2-4 & 2 & 0 \\ 1 & 2-3 & 0 \\ 0 & 0 & 2-2 \end{bmatrix} = \mathrm{NUL}\begin{bmatrix} -2 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \mathrm{NUL}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{SPAN}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$\left(x - y = 0 \implies x = y \implies \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$\underline{\lambda=5}$ $$\mathrm{NUL}(5I-A) = \mathrm{NUL}\begin{bmatrix} 5-4 & 2 & 0 \\ 1 & 5-3 & 0 \\ 0 & 0 & 5-2 \end{bmatrix} = \mathrm{NUL}\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \mathrm{NUL}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{SPAN}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$$\left(\begin{matrix} x + 2y = 0 \\ z = 0 \end{matrix} \implies \begin{matrix} x = -2y \\ z = 0 \end{matrix} \implies \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right)$$

3) SEE ABOVE

4. *(10 points)* Use the formula for inverses using determinants (section 3.3) to calculate A^{-1}, where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

1) $$\operatorname{DET}(A) = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= -6 + 1 - 6 - 2 + 4 + 8$$

$$= -1$$

2) $$A^{-1} = \frac{1}{|A|} \begin{bmatrix} +\begin{vmatrix} -2 & -1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \end{bmatrix}^T$$

$$= - \begin{bmatrix} -5 & -8 & +6 \\ -1 & -1 & 1 \\ 3 & 5 & -4 \end{bmatrix}^T$$

$$= - \begin{bmatrix} -5 & -1 & 3 \\ -8 & -1 & 5 \\ +6 & 1 & -4 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 5 & 1 & -3 \\ 8 & 1 & -5 \\ -6 & -1 & 4 \end{bmatrix}}$$

3. *(10 = 5 + 5 points)*
   (a) Is the following matrix A diagonalizable? Why or why not?
   (b) Find the $\mathcal{B}$–matrix of A, where $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 6 \end{bmatrix}$$

(a) $|\lambda I - A| = \begin{vmatrix} \lambda-4 & -1 \\ 1 & \lambda-6 \end{vmatrix} = (\lambda-4)(\lambda-6)+1$

$= \lambda^2 - 10\lambda + 24 + 1$

$= \lambda^2 - 10\lambda + 25$

$= (\lambda-5)^2 = 0 \quad \Rightarrow \underline{\lambda = 5}$

$\underline{\lambda = 5}$ $\quad NUL(5I - A) = NUL\begin{bmatrix} 5-4 & -1 \\ 1 & 5-6 \end{bmatrix} = NUL\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = NUL\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$= SPAN\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\left(x - y = 0 \Rightarrow x = y \Rightarrow \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

ONLY 1 (L.I.) EIGENVECTOR, SO NO

(b) $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$

$$(x-1)\left(\left[\begin{array}{cc|cc} 1 & 1 & 5 & 6 \\ 1 & 2 & 5 & 11 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 5 & 6 \\ 0 & 1 & 0 & 5 \end{array}\right]\right)(x-1) \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 5 & 1 \\ 0 & 1 & 0 & 5 \end{array}\right]$$

$\quad P \quad AP$

$$\boxed{B = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}}$$

5. *(10 = 4 + 1 + 2 + 1 + 2 points)* In this question, there will be no partial credit for each sub-part.
For the following matrix A, find:
(a) A basis for $Nul(A)$
(b) $\dim(Nul(A))$
(c) A basis for $Col(A)$
(d) $Rank(A)$
(e) State the Rank Theorem

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \sim B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} (x-2) \\ (x1) \\ \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 2 & -3 & 6 & 0 & -1 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (x-2)$$

$$\sim \begin{bmatrix} 2 & -3 & 0 & 0 & -9 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} & y & & & s \end{array}$$
$$\sim \begin{bmatrix} 1 & -3/2 & 0 & 0 & -9/2 \\ 0 & 0 & 1 & 0 & 4/3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{RREF}$$

(a)
$$\begin{cases} x - 3/2\,y - 9/2\,s = 0 \\ z + 4/3\,s = 0 \\ t + 3s = 0 \end{cases} \Rightarrow \begin{cases} x = 3/2\,y + 9/2\,s \\ z = -4/3\,s \\ t = -3s \end{cases}$$

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \end{bmatrix} = \begin{bmatrix} 3/2\,y + 9/2\,s \\ y \\ -4/3\,s \\ -3s \\ s \end{bmatrix} = y\begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix}$$

<u>BASIS FOR NUL(A)</u> $\left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\}$

(d) $RANK(A) = 3$

(b) $DIM(NUL(A)) = 2$

(e) $DIM(NUL(A)) + RANK(A) = N$

(c) <u>BASIS FOR COL(A)</u> $\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\}$

6. *(10 points)* Use the Gram-Schmidt process to find the QR decomposition of A, where

$$A = \begin{array}{c} \begin{matrix} U_1 & U_2 & U_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

1) $V_1 = U_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\hat{U}_2 = \left(\frac{U_2 \cdot V_1}{V_1 \cdot V_1}\right) V_1 = \left(\frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}\right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$$V_2 = U_2 - \hat{U}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

CHECK $\quad V_1 \cdot V_2 = 1 - 1 + 0 = 0$ ✓

$$\hat{U}_3 = \left(\frac{U_3 \cdot V_1}{V_1 \cdot V_1}\right) V_1 + \left(\frac{U_3 \cdot V_2}{V_2 \cdot V_2}\right) V_2 = \left(\frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}\right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \left(\frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}\right) \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$V_3 = U_3 - \hat{U}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

CHECK $\quad V_3 \cdot V_1 = -1 + 1 + 0 = 0$ ✓ $, \quad V_3 \cdot V_2 = -1 - 1 + 2 = 0$

2) $W_1 = \dfrac{V_1}{\|V_1\|} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$W_2 = \dfrac{V_2}{\|V_2\|} = \dfrac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$

$W_3 = \dfrac{V_3}{\|V_3\|} = \dfrac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$$\boxed{Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}}$$

3) $R = Q^T A$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\boxed{R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}}$$

7. *(10 points)* Find the least-squares solution and the least-squares error of $Ax = b$. You may use any method taught in this course.

$A =$ [redacted], $b =$ [redacted]

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

1) $A^T A \hat{x} = A^T b$

$$\begin{bmatrix} 2 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \hat{x} = \begin{bmatrix} 2 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 6 \\ 2 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$(\div 6) \left[\begin{array}{cc|c} 12 & 6 & 6 \\ 2 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 1 \\ \cancel{2} & \cancel{2} & \cancel{1} \end{array}\right] (\times -1)$$

$$\to \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right] (\times -1)$$

$$\to \left[\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 0 \end{array}\right].$$

$$\boxed{\hat{x} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}$$

2) Error: $\|A\hat{x} - b\| = \left\| \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\|$

$$= \left\| \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\| = \sqrt{3}$$

8. *(10 points, 5 points each)* Label each statement as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer, meaning that if the statement is true, you have to explain why it's true, and if the statement is false, you have to give an explicit counterexample and show why it's a counterexample.

TRUE

(a) If A is similar to B and B is diagonalizable, then A is diagonalizable.

$A \sim B \Rightarrow A = PBP^{-1}$

B DIAGONALIZABLE $\Rightarrow B = QDQ^{-1}$ FOR D DIAGONAL

THEN $A = PBP^{-1} = P(QDQ^{-1})P^{-1}$

$= (PQ)D(Q^{-1}P^{-1})$

$= (PQ)D(PQ)^{-1}$

$= RDR^{-1}$, $R = PQ$, D = DIAGONAL

SO A IS DIAGONALIZABLE

TRUE

(b) For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, we have:

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

$\|U+V\|^2 + \|U-V\|^2 = (U+V)\cdot(U+V) + (U-V)\cdot(U-V)$

$= U\cdot U + \cancel{U\cdot V} + \cancel{V\cdot U} + V\cdot V$

$+ U\cdot U - \cancel{U\cdot V} - \cancel{V\cdot U} + V\cdot V$

$= 2U\cdot U + 2V\cdot V$

$= 2\|U\|^2 + 2\|V\|^2$

9. *(10 points)* Suppose A satisfies $A^T = A$ and let **u** be an eigenvector of A corresponding to λ and **v** an eigenvector of A corresponding to μ, where $\lambda \neq \mu$. Show that **u** and **v** are orthogonal. EXPLAIN WHERE YOU USED $\lambda \neq \mu$

   Hint: Calculate $(A\mathbf{u}) \cdot \mathbf{v}$ in two different ways. You may want to use that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$.

ON THE ONE HAND:

$$(AU)\cdot V = (\lambda U)\cdot V = \lambda(U\cdot V)$$

ON THE OTHER HAND:

$$(AU)\cdot V = (AU)^T V = U^T A^T V$$

$$= U^T A V \qquad (A^T = A)$$

$$= U\cdot(AV)$$

$$= U\cdot(\mu V)$$

$$= \mu(U\cdot V)$$

SO $\lambda(U\cdot V) = \mu(U\cdot V)$

$\Rightarrow \underbrace{(\lambda-\mu)}_{\neq 0}(U\cdot V) = 0$

$\Rightarrow U\cdot V = \dfrac{0}{\lambda-\mu} = 0$

$\Rightarrow U\cdot V = 0$

$\Rightarrow U \perp V$

10. *(10 points)* The Grand Finale!!!

WEREWOLVES

Welcome to the final Twilight battle between Team Edward (vampires) and Team Jacob ([redacted]). Assume that the number of vampires v_n and the number of [redacted] after each round n are related by the following system WEREWOLVES

$$\begin{cases} v_{n+1} = 2v_n - \frac{1}{2}\,[redacted]\ W_N \\ z_{n+1} = 3v_n - \frac{1}{2}\,[redacted]\ W_N \end{cases}$$

Find all the initial values $\begin{bmatrix} v_0 \\ W_0 \end{bmatrix}$ such that we have $\begin{bmatrix} v_\infty \\ W_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (which means in the long-run, both species die out).

Note: Unlike the Pokemon battle-example, assume here that negative values of v_n and [redacted] are allowed

W_N

1) $$\begin{bmatrix} V_{N+1} \\ W_{N+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1/2 \\ 3 & -1/2 \end{bmatrix}}_{A} \begin{bmatrix} V_N \\ W_N \end{bmatrix}$$

2) DIAGONALIZE A

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 1/2 \\ -3 & \lambda + \frac{1}{2} \end{vmatrix} = (\lambda - 2)\left(\lambda + \frac{1}{2}\right) + \frac{3}{2}$$

$$= \lambda^2 - \frac{3}{2}\lambda - 1 + \frac{3}{2} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}$$

$$= (\lambda - 1)\left(\lambda - \frac{1}{2}\right) = 0 \Rightarrow \underline{\lambda = 1, \frac{1}{2}}$$

$\underline{\lambda = 1}$ $$\text{NUL}(I - A) = \text{NUL}\begin{bmatrix} 1-2 & 1/2 \\ -3 & 1 + \frac{1}{2} \end{bmatrix} = \text{NUL}\begin{bmatrix} -1 & 1/2 \\ -3 & 3/2 \end{bmatrix} = \text{NUL}\begin{bmatrix} 1 & -1/2 \\ 1 & -1/2 \end{bmatrix}$$

$$= \text{NUL}\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} = \text{SPAN}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

$$x - \frac{1}{2}y = 0 \Rightarrow x = \frac{1}{2}y \Rightarrow \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2\, y \\ y \end{bmatrix} = y\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \overset{(\times 2)}{\sim} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\underline{\lambda = \frac{1}{2}}$ $\quad NUL\left(\frac{1}{2}I - A\right) = NUL\begin{bmatrix} 1/2 - 2 & 1/2 \\ -3 & 1/2 + 1/2 \end{bmatrix}$

$= NUL\begin{bmatrix} -3/2 & 1/2 \\ -3 & 1 \end{bmatrix} = NUL\begin{bmatrix} 1 & -1/3 \\ 1 & -1/3 \end{bmatrix} = NUL\begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$

$= SPAN\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$

$\left(x - \frac{1}{3}y = 0 \Rightarrow x = \frac{1}{3}y \quad \Rightarrow \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3\,y \\ y \end{bmatrix} = y\begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)$

3) SO $\quad A = PDP^{-1}, \quad P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

SO $\quad A^N = PD^NP^{-1}$

$A^\infty = PD^\infty P^{-1}$

$= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1^\infty & 0 \\ 0 & \left(\frac{1}{2}\right)^\infty \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$ $\qquad\qquad \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$

WE NEED TO FIND $\begin{bmatrix} V_0 \\ W_0 \end{bmatrix}$ SUCH THAT $\begin{bmatrix} V_\infty \\ W_\infty \end{bmatrix} = A^\infty\begin{bmatrix} V_0 \\ W_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

THAT IS $\quad \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} V_0 \\ W_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 6 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left\{ x - \tfrac{1}{3} y = 0 \Rightarrow x = \tfrac{1}{3} y \Rightarrow \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3\, y \\ y \end{bmatrix} = y \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$

<u>ANS</u> $\boxed{\text{span}\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}}$