MATH 3A – FINAL EXAM

Name: ________________________

Student ID: ________________________

Instructions: This is it, your final hurdle to freedom! Welcome to your Final Exam! You have 120 minutes to take this exam, for a total of 100 points. No books, notes, calculators, or cellphones are allowed. Remember that you are not only graded on your final answer, but also on your work. If you need to continue your work on the back of the page, clearly indicate so, or else your work will be discarded. May your luck be orthonormal! :)

Academic Honesty Statement: I hereby certify that the exam was taken by the person named and without any form of assistance and acknowledge that any form of cheating (no matter how small) results in an automatic F in the course, and will be further subject to disciplinary consequences, pursuant to section 102.1 of the UCI Student Code of Conduct.

Signature: __________________________________

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Date: Friday, March 22, 2019.

1. *(10 points, 1 point each)* Label each statement as **TRUE** or **FALSE**. In this question, you do **NOT** have to justify your answer. Each correct answer will get 1 point and each incorrect or illegible answer will get 0 points.

   (a) If A is similar to I (the identity matrix), then $A = I$

   (b) If A is invertible and λ is an eigenvalue of A, then $\frac{1}{\lambda}$ must be an eigenvalue of A^{-1}

   (c) For any matrix A, $Col(A)$ is orthogonal to $Nul(A)$

   (d) A 3×3 matrix with eigenvalues $\lambda = 0$ and $\lambda = 1$ can never be diagonalizable

   (e) A 3×3 matrix with eigenvalues $\lambda = 0$ and $\lambda = 1$ can never be invertible

   (f) If A is a 2×3 matrix, then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is never one-to-one

   (g) If A is a 2×3 matrix, then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is never onto $\mathbb{R}^2$

   (h) The least-squares solution $\widehat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ is a vector in $\mathbb{R}^n$ that satisfies $\|A\mathbf{x} - \mathbf{b}\| \leq \|A\widehat{\mathbf{x}} - \mathbf{b}\|$ for all $\mathbf{x}$

   (i) If $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A corresponding $\lambda = 1$ and $\lambda = -1$ respectively, then $A^{2019}(\mathbf{u} - \mathbf{v}) = \mathbf{u} + \mathbf{v}$

   (j) For any square matrix A, $\det(3A) = 3\det(A)$

2. *(10 points)* Find a diagonal matrix D and an invertible matrix P with $A = PDP^{-1}$, where

$$A = \begin{bmatrix} 4 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

3. *(10 = 5 + 5 points)*
   (a) Is the following matrix A diagonalizable? Why or why not?
   (b) Find the $\mathcal{B}-$matrix of A, where $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 6 \end{bmatrix}$$

4. *(10 points)* Use the formula for inverses using determinants (section 3.3) to calculate A^{-1}, where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

5. *(10 = 2 + 2 + 1 + 2 + 1 + 2 points)* In this question, there will be no partial credit for each sub-part.

For the following matrix A, find:

(a) The reduced row-echelon form (RREF) of A
(b) A basis for $Nul(A)$
(c) $\dim(Nul(A))$
(d) A basis for $Col(A)$
(e) $Rank(A)$
(f) State the Rank Theorem

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \sim B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6. *(10 points)* Use the Gram-Schmidt process to find the QR decomposition of A, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

7. *(10 points)* Find the least-squares solution **and** the least-squares error of $A\mathbf{x} = \mathbf{b}$. You may use any method taught in this course.

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

8. *(10 points, 5 points each)* Label each statement as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer, meaning that if the statement is true, you have to explain why it's true, and if the statement is false, you have to give an explicit counterexample and show why it's a counterexample.

   (a) If A is similar to B and B is diagonalizable, then A is diagonalizable.

   (b) For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, we have:

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

9. *(10 points)* Suppose A satisfies $A^T = A$ and let $\mathbf{u}$ and $\mathbf{v}$ be eigenvectors of A corresponding to λ and μ respectively, where $\lambda \neq \mu$. Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Explain where you used the fact that $\lambda \neq \mu$.

   Hint: Calculate $(A\mathbf{u}) \cdot \mathbf{v}$ in two different ways. You may want to use that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$.

10. *(10 points)* The Grand Finale!!!

Welcome to the final Twilight battle between Team Edward (vampires) and Team Jacob (werewolves). Assume that the number of vampires v_n and the number of werewolves w_n after each round n are related by the following system

$$\begin{cases} v_{n+1} = 2v_n - \frac{1}{2}w_n \\ w_{n+1} = 3v_n - \frac{1}{2}w_n \end{cases}$$

Find all the initial values $\begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$ such that we have $\begin{bmatrix} v_\infty \\ w_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (which means in the long-run, both species die out).

Note: Unlike the Pokemon battle-example, assume that negative values of v_n and w_n are allowed

(Scratch paper)