WEDNESDAY, APRIL 3, 2019

LECTURE 2 - SUBSPACES (SECTION 1.3)

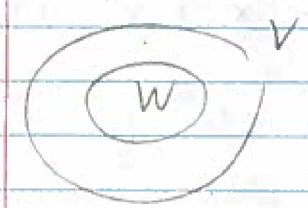
PREVIOUSLY ON PEYAM BATTLESTAR GALACTICA, WE DISCOVERED THE NOTION OF A VS, WHICH IS THIS FANTASTIC CONCEPT THAT CAPTURES THE ESSENCE OF $\mathbb{R}^n$, AND WE'VE SEEN THAT A LOT OF THINGS ARE VS.

PROBLEM IT'S A PAIN TO CHECK THAT SOMETHING IS A VS!
10 CONDITIONS! AIN'T NOBODY GOT TIME FOR THAT!!!
LUCKILY, IT TURNS OUT THERE'S A QUICKER WAY TO CHECK SOMETHING IS A VS

I - DEFINITION(S)

NAMELY, JUST SHOW IT'S A SUBSPACE OF ANOTHER VS

DEF: LET W BE A SUBSET OF V (= VECTOR SPACE)
THEN W IS A SUBSPACE OF V IF W IS ITSELF A VS



(SPECIAL KIND OF SUBSET OF V, ONE THAT IS ITSELF A VS)

ANALOGY IF V = ALL OF REDDIT, W = SUBREDDIT
(SUBSET OF REDDIT THAT IS ITSELF A REDDIT)

NOW THIS IS NOT VERY PRACTICAL, IT DOESN'T REALLY TELL YOU HOW TO KNOW SOMETHING IS A SUBSPACE. LUCKILY THERE'S A BETTER DEF:

CLAIM: W IS A SUBSPACE OF V IF AND ONLY IF:

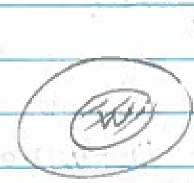
(a) 0 IS IN W (0 = ZERO VECTOR OF V)
(b) IF x AND y ARE IN W, $x+y$ ARE IN W ("CLOSED UNDER +
(c) IF x IS IN W AND c IS IN F, cx IS IN W ("CLOSED UNDER ·")

WHAT IS THIS SAYING? TO SHOW SOMETHING IS A SUBSPACE,
DON'T NEED ALL 10 PROPERTIES, ONLY NEED 3
(BUY 3 GET 7 FREE)

WHY? ($\Leftarrow$) SUPPOSE (a)−(c) HOLD, SHOW W IS A VS
NEED TO CHECK (1)−(10) (FROM LAST TIME)

POINT WE ACTUALLY DON'T HAVE TO! MOST OF THEM COME FOR FREE

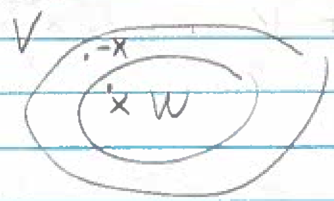
EX CHECK $x+y=y+x$



SINCE $\forall x,y \in V$, $x+y=y+x$

$\forall x,y \in W$, $x+y=y+x$ (WE SAY W INHERITS THE PROPERTIES FROM V)

ONLY ISSUE IF $x \in W$, WE ONLY KNOW THAT $-x$ IS IN V



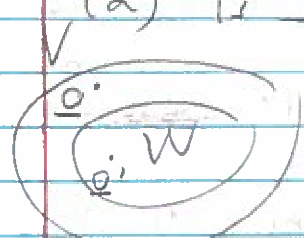
BUT $-x = \underset{c}{(-1)}x \in W$ (BY (c))

SO $-x$ IS INDEED IN W

($\Rightarrow$) SUPPOSE W IS A VS, SHOW (a)−(c) HOLD

(b) IF x & y ARE IN W, THEN $x+y$ IS IN W (B/C W IS A VS)
(c) SIMILAR

(a) IS $\underline{0}$ IN W? ($\underline{0}$ = ZERO VECTOR IN V)



ON THE ONE HAND: $x+\underline{0}=x \ \forall x \in V$
SO $x+\underline{0}=x \ \forall x \in W$
BUT SINCE W IS A VS, IT HAS ITS OWN ZERO VECTOR $\underline{0}'$
WITH $x+\underline{0}'=x \ \forall x \in W$

THEN $\cancel{x} + \underline{0} = \cancel{x} + \underline{0}' \Rightarrow \underline{0} = \underline{0}' \in W$

SO $\underline{0} \in W$ ∎

II – EXAMPLES (AND NON-EXAMPLES)

EX $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \text{ IN } M_{2\times2} \mid a, b, c \in \mathbb{R} \right\}$

(UPPER-TRIANGULAR MATRICES)

NOTE HERE $V = M_{2\times2}$

(a) IS $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ IN W? YES $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $a, b, c = 0$ ✓

(b) IF $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ AND $\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}$ ARE IN W, THEN

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ 0 & c+c' \end{bmatrix} \text{ IS IN } W \checkmark$$

(c) IF $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ IS IN W AND $\tilde{c}$ IS IN $\mathbb{F}$, THEN

$$\tilde{c} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} \tilde{c}a & \tilde{c}b \\ 0 & \tilde{c}c \end{bmatrix} \text{ IS IN } W \checkmark$$

EX $W =$ POLYNOMIALS OF DEGREE EXACTLY 2

NO $\underline{0}$ IS NOT IN IT

(OR $\underbrace{x^2}_{\text{IN } W} + \underbrace{(-x^2+1)}_{\text{IN } W} = 1 \leftarrow$ NOT IN W

EX $W = \{ (x,y) \text{ IN } \mathbb{R}^2 \mid x^2 + y^2 = 0 \}$

YES $W = \{ (0,0) \}$

FACT $\{\underline{0}\}$ AND V ARE ALWAYS SUBSPACES OF V

V = SPACE OF DIFFERENTIABLE FUNCTIONS FROM $\mathbb{R}$ TO $\mathbb{R}$ $(C^\infty(\mathbb{R}))$

EX $W = \{ y \text{ IN } V \mid y'' - 5y' + 6y = 0 \}$

(a) 0 IN W? $0'' - 5(0') + 6(0) = 0$? YES

(b) SUPPOSE y_1 & y_2 ARE IN W, SO $y_1'' - 5y_1' + 6y_1 = 0$

$y_2'' - 5y_2' + 6y_2 = 0$

SHOW $y_1 + y_2$ IS IN W:

$$(y_1 + y_2)'' - 5(y_1 + y_2)' + 6(y_1 + y_2) \stackrel{?}{=} 0$$

$$= y_1'' + y_2'' - 5y_1' - 5y_2' + 6y_1 + 6y_2$$

$$= (y_1'' - 5y_1' + 6y_1) + (y_2'' - 5y_2' + 6y_2) = 0 + 0 = 0 \checkmark$$

(c) SIMILAR

POINT CAN WRITE DIFFERENTIAL EQUATIONS IN TERMS OF LINEAR ALGEBRA

SOLUTIONS TO (LINEAR) DIFFERENTIAL EQUATIONS ARE SUBSPACES.

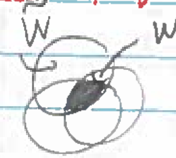
(EVEN THOUGH WE DON'T KNOW WHAT THE SOLUTIONS ARE,

WE KNOW THEY FORM A SUBSPACE)

III – Unions and Intersections

Since subspaces are so great, you may ask: How can we create new subspaces from old ones? We'll discuss this more next time when we talk about span, but for now, there's a neat way of forming subspaces, and it lies @ the intersection of happiness and linear algebra!!

Theorem The intersection of any number of subspaces of V is a subspace of V.



Why? Let $\mathcal{E}$ be a family of subspaces of V and W be their intersection.

(a) Is 0 in W?

For all $\tilde{W}$ in $\mathcal{E}$, since $\tilde{W}$ is a subspace of V, $0 \in \tilde{W}$

So 0 is in W (by def of intersection) ✓

(b) Suppose x & y are in W, then $\forall \tilde{W} \in \mathcal{E}$,

x & y are in $\tilde{W}$ (def of $\cap$)

But since $\tilde{W}$ is a subspace of V, $x+y$ is in $\tilde{W}$ ✓

Since $\tilde{W}$ was arbitrary, $x+y \in W$ (def of $\cap$)

(c) If x is in W and c is in $\mathbb{F}$, then $\forall \tilde{W} \in \mathcal{E}$,

x is in $\tilde{W}$, so cx is in $\tilde{W}$ (since $\tilde{W}$ is a subspace of V)

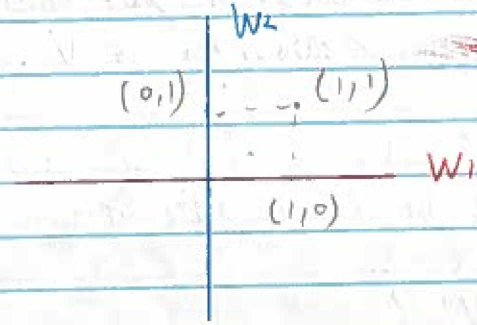
Since $\tilde{W}$ is arbitrary, cx is in W ✓

HENCE W IS A SUBSPACE OF V ∎

NOTE IN GENERAL, THE UNION OF SUBSPACES IS NOT A SUBSPACE!

EX LET $W_1 =$ x-AXIS IN $\mathbb{R}^2$,

$W_2 =$ y-AXIS IN $\mathbb{R}^2$



THEN $W_1 \cup W_2 =$ x-AXIS AND y-AXIS

NOT A SUBSPACE, BECAUSE $(1,0) \in W_1 \cup W_2$ (SINCE $(1,0) \in W_1$)
AND $(0,1) \in W_1 \cup W_2$ (SINCE $(0,1) \in W_2$)

BUT $(1,0) + (0,1) = (1,1)$ NOT IN $W_1 \cup W_2$

NOTE $W_1 \cap W_2 = \{(0,0)\}$, SUBSPACE OF $\mathbb{R}^2$.