Previously on Peyam Hatlesian Galactica, we discovered the notion of a \( V \), which is this fantastic concept that captured the essence of \( \mathbb{R}^n \), and we've seen that a lot of things are \( V \)s.

**Problem:** It's a pain to check that something is a \( V \)!

**10 Conditions:** Ain't nobody got time for that!!!

**Luckily, it turns out there's a quicker way to check something is a \( V \)\( \text{\textperiodcentered} \)**

**1. Definition(s)**

Namely, just show it's a subspace of another \( V \)

**Def:** Let \( W \) be a subset of \( V \) (= vector space)

Then \( W \) is a subspace of \( V \) if \( W \) is itself a \( V \)

\[ \text{(special kind of subset of } V \text{, one that is itself a } V) \]

**Analysis:** If \( V = \text{all of } \text{r.e.d.e.d} \), \( W = \text{subset of } \text{r.e.d.e.d that is itself a } \text{r.e.d.e.d} \)

Now this is not very practical; it doesn't really tell you how to know something is a subspace. Luckily there's a better def:

**Claim:** \( W \) is a subspace of \( V \) if and only if:

1. \( \emptyset \) is in \( W \) \( (\emptyset = \text{empty vector of } V) \)
2. If \( x \) and \( y \) are in \( W \), \( x + y \) are in \( W \) ("closed under")
3. If \( x \) is in \( W \) and \( c \) is in \( F \), \( cx \) is in \( W \) ("closed under")
What is this saying? To show something is a subspace, don't need all 10 properties, only need 3 (buy 3 get 7 free)

WHY? \( \implies \) suppose (a)-(c) hold, then \( W \) is a vs

Need to check (1)-(10) (from last time)

Point: we actually don't have to! Most of these come for free.

Ex: check \( x + y = y + x \)

Since \( x, y \in V \), \( x + y = y + x \)

\[ \forall x, y \in W, x + y = y + x \] (we say \( W \) inherits the properties from \( V \))

Only issue: if \( x \in W \), we only know that \(-x \) is in \( V \)

\[ \begin{array}{c}
\forall x \in W \\
\exists y \in W
\end{array} \]

But \( -x = (-1)x \in W \) (by (c))

So \(-x \) is indeed in \( W \)

\( \implies \) suppose \( W \) is a vs, show (a)-(c) hold

(a) If \( x, y \) are in \( W \), then \( x + y \) is in \( W \) (\( \forall x, y \in W \\Rightarrow x + y \in W \))

(c) simlarly

(b) \( 0 \) in \( W \)? \( (0 = \) zero vector in \( V ) \)

\[ \begin{array}{c}
\forall x \in W \\
\exists y \in W
\end{array} \]

On the one hand: \( x + 0 = x \neq x \in V \)

\[ \exists x \in W \]

But since \( W \) is a vs, it has its own zero vector with \( x + 0' = x \neq x \in W \)
THEN \[ x + a = x + a' \Rightarrow a = a' \in W \]
so \( a \in W \).

**II - Example: (and non-example)**

**EX**

\[ W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | \text{in } M_{2x2}, a, b, c \in \mathbb{R} \right\} \]

(Upper - triangular matrices)

**Note** Here \( V = M_{2x2} \)

(a) is \( \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \) in \( W \): \( \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \), \( a, b, c \in \mathbb{R} \)

(b) if \( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \) and \( \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} \) are in \( W \), then

\[
\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} a + a' & b + b' \\ 0 & c + c' \end{bmatrix} \text{ is in } W
\]

(c) if \( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \) is in \( W \) and \( c \) is in \( F \), then

\[
c \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ca & cb \\ 0 & cc \end{bmatrix} \text{ is in } W
\]

**EX**

\( W = \text{ polynomials of degree exactly 2} \)

No \( x + \text{ not in } W \)

Let \( x^2 + (-x^2 + 1) = 1 \) \( \not\in \text{ not in } W \)
\[ \text{EX} \quad W = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 0 \} \]

\[ \text{YES} \quad W = \{ (0, 0) \} \]

**Fact**  
\{ 0 \} and \ V \ \text{are always subspaces of} \ V

\[ V = \text{space of differentiable functions from} \ \mathbb{R} \ \text{to} \ \mathbb{R} \quad (\mathcal{C}(\mathbb{R})) \]

\[ \text{EX} \quad W = \{ y'' - 5y' + 6y = 0 \} \]

(a) \ \text{0 in} \ W \quad 0'' - 5(0') + 6(0) = 0 \quad \text{YES}

(b) \ \text{Suppose} \ y_1 \ \text{and} \ y_2 \ \text{in} \ W, \ \text{so} \ y_1'' - 5y_1' + 6y_1 = 0 \quad y_2'' - 5y_2' + 6y_2 = 0

\text{Show} \ y_1 + y_2 \ \text{is in} \ W:

\[ (y_1 + y_2)'' - 5(y_1 + y_2)' + 6(y_1 + y_2) = 0 \]

\[ = y_1'' + y_2'' - 5(y_1' + y_2') + 6y_1 + 6y_2 \]

\[ = (y_1'' - 5y_1' + 6y_1) + (y_2'' - 5y_2' + 6y_2) = 0 + 0 = 0 \quad \text{YES} \]

(c) \ \text{Similar}

\[ \text{Point} \quad \text{can write differential equation in terms of linear algebra} \]

\[ \text{solution to (linear) differential equation are subspaces} \]

\[ \text{(even though we don't know what the solution are, we know they form a subspace)} \]
III. UNION AND INTERSECTION

Since subspaces are so great, you may ask: how can we create new subspaces from old ones? We'll discuss this more next time when we talk about span, but for now, there's a neat way of forming subspaces, and it lies in the intersection of linear sets and linear algebra.

THEOREM: The intersection of any number of subspaces of $V$ is a subspace of $V$.

Why? Let $F$ be a family of subspaces of $V$ and $W$ be their intersection.

(a) $0 \in W$?

For all $W \in F$, since $W$ is a subspace of $V$, $0 \in W$.

So $0 \in W$ (by def. of intersection).

(b) Suppose $x, y \in W$, then $x + y \in F$.

$x, y \in W$ (def. of $W$).

Since $W$ is a subspace of $V$, $x + y \in W$.

Since $W$ was arbitrary, $x + y \in W$ (def. of $F$).

(c) If $x \in W$ and $c \in W$, then $cx \in W$.

$x \in W$ so $cx \in W$ (since $W$ is a subspace of $V$.

Since $W$ is arbitrary, $cx \in W$. 
Hence $W$ is a subspace of $V$.

Note: in general, the union of subspaces is not a subspace!

**Example:** Let $W_1 = x-ax_1$ in $m^1$,

$W_2 = y-ax_1$ in $m^2$

Then $W_1 \cup W_2 = x-ax_1$ and $y-ax_1$

Not a subspace, because $(1,0) \in W_1 \cup W_2$ (since $(1,0) \in W_1$) and $(0,1) \in W_1 \cup W_2$ (since $(0,1) \in W_2$)

$W_1 \cup W_2 = \{(1,1)\} \not\subset$ in $W_1 \cup W_2$

Note: $W_1 \cap W_2 = \{0,0\}$, subspace of $m^2$. 