AP SOLUTIONS (HW #1)

First of all, if \( x \) is in \( V = \mathbb{R} \) and \( y \) is in \( V \), then
\[
x \oplus y = x + y - 1, \quad \text{which is in } V
\]

And if \( x \) is in \( V \) and \( c \) is in \( M \), then
\[
c \circ x = cx + (1-c), \quad \text{which is in } V
\]

\( \text{(Vs 1)} \) Suppose \( x \) and \( y \) are in \( V \), then
\[
x \oplus y = x + y - 1 = y + x - 1 = y \oplus x
\]

\( \text{(Vs 2)} \) Suppose \( x, y, z \) are in \( V \), then
\[
(x \oplus y) \oplus z = (x + y - 1) \oplus z
\]
\[
= (x + y - 1) + z - 1
\]
\[
= x + y + z - 2
\]

And \( x \oplus (y \oplus z) = x + (y + z - 1) - 1 \)
\[
= x + y + z - 2
\]

So \( (x \oplus y) \oplus z = x \oplus (y \oplus z) \)

\( \text{(Vs 3)} \) [PREP LER! TRY TO FIGURE OUT WHAT THE ZERO VECTOR IS!]

Let \( x \) be in \( V \), then:
\[ x \oplus 0 = x \Rightarrow x + 0 - 1 = x = 0 = 1 \]

So the 0 vector in \( V \) is just 1, how cool is that!

Let \( x \) be in \( V \), then

\[ x \oplus 0 = x \oplus 1 = (x + 1) - 1 = x \]

(V. 4) Let's find \(-x\):

Let \( x \) be in \( V \), then \( x \oplus y = 0 \)

\[ x + y = 0 \Rightarrow x + y - 1 = 1 \Rightarrow x + y = 2 \Rightarrow y = 2 - x \]

so \( -x = 2 - x \)

Let \( x \) be in \( V \), and \( y = 2 - x \) in \( V \), then

\[ x \oplus y = x + y - 1 = x + 2 - x - 1 = 1 = 0, \text{ so } x \oplus y = 0 \]

(V.5) Let \( x \) be in \( V \), then

\[ x \oplus x = 4x + (1 - 1) = x \]

(V.6) Let \( a, b \) in \( M \) and \( x \) in \( V \), then

\[ (ab) \oplus x = abx + (1 - ab) \]

And \( a \oplus (b \oplus x) = a(b \oplus x) + 1 - a \)

\[ = a(bx + 1 - b) + 1 - a \]

\[ = abx + a(1 - b) + 1 - a \]

\[ = abx + 2 - ab + 1 - x = abx + (1 - ab) = (ab) \oplus x \]
\[(V5 \; 7) \text{ Let } a \in M \text{ and } x, y \in V, \text{ then} \]

\[
\begin{align*}
    a \circ (x \oplus y) &= a(x \oplus y) + (1 - a) \\
    &= a(x + y - 1) + 1 - a \\
    &= ax + ay - a + 1 - a \\
    &= ax + ay + 1 - 2a
\end{align*}
\]

\[
\begin{align*}
    (a \circ x) \oplus (a \circ y) &= (a \circ x) + (a \circ y) - 1 \\
    &= ax + (1 - a) + ay + (1 - a) - 1 \\
    &= ax + ay + 1 - 2a = a \circ (x \oplus y)
\end{align*}
\]

\[(V5 \; 8) \text{ Let } a, b \in M \text{ and } x \in V, \text{ then} \]

\[
\begin{align*}
    (a + b) \circ x &= (a + b) x + (1 - (a + b)) \\
    &= ax + bx + 1 - a - b
\end{align*}
\]

\[
\begin{align*}
    (a \circ x) \oplus (b \circ x) &= (a \circ x) + (b \circ x) - 1 \\
    &= ax + (1 - a) + bx + (1 - b) - 1 \\
    &= ax + bx + 1 - a - b \\
    &= (a + b) \circ x
\end{align*}
\]

Hence, \( V \) is a vector space over \( M \).