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FRIDAY, APRIL 5, 2019

LECTURE 3 – LINEAR COMBINATIONS AND SPAN (SECTION 1.4)

HAPPY F, AND TODAY WE'LL DISCOVER A NEAT WAY OF COMBINING VECTORS, WHICH NATURALLY LEADS TO THE CONCEPT OF A LINEAR COMBINATION

I – LINEAR COMBINATIONS

DEF $x \in V$ IS A LINEAR COMBINATION OF $u_1, \dots, u_n \in V$ IF

$$x = a_1 u_1 + \dots + a_n u_n \quad \text{FOR SOME } a_1, \dots, a_n \in \mathbb{F}$$

EX $x = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ IS A LINEAR COMBO OF $u_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ AND $u_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

BECAUSE $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$x = a_1 u_1 + a_2 u_2$$

EX IS $x = (0,-2,4)$ A LINEAR COMBO OF $u_1 = (1,1,3)$, $u_2 = (2,-1,0)$, $u_3 = (1,3,-1)$?

ARE THERE a_1, a_2, a_3 WITH

$$x = a_1 u_1 + a_2 u_2 + a_3 u_3 \ ?$$

$$(0,-2,4) \overset{?}{=} a_1(1,1,3) + a_2(2,-1,0) + a_3(1,3,-1)$$

$$(0,-2,4) \overset{?}{=} (a_1 + 2a_2 + a_3,\ a_1 - a_2 + 3a_3,\ 3a_1 - a_3)$$

$$\begin{cases} a_1 + 2a_2 + a_3 = 0 \\ a_1 - a_2 + 3a_3 = -2 \\ 3a_1 - a_3 = 4 \end{cases} \qquad \text{SYSTEM OF EQUATIONS}$$

SOLVE THIS (USING ROW-REDUCTION – SEE PAGE 7, OR SEE TECHNIQUE IN THE BOOK)

$$\Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 0 \\ a_3 = -1 \end{cases}$$

$\Rightarrow \quad (0,-2,4) = 1\,(1,1,3) + 0\,(2,-1,0) + (-1)\,(1,3,-1)$ YES ✓

NOW, OF COURSE, ONCE YOU'VE TAKEN ONE LINEAR COMBO, YOU MAY ASK: WHAT ABOUT ALL THE POSSIBLE LINEAR COMBINATIONS?
AND THIS IS INDEED USEFUL AND HAS ITS OWN NAME:

II - SPAN

DEF IF S IS ANY SUBSET OF V, THEN $\text{SPAN}(S)$ IS THE SET OF ALL (FINITE) LINEAR COMBINATIONS OF VECTORS IN V

$x \in \text{SPAN}(S) \iff x = a_1 u_1 + \dots + a_N u_N$ FOR SOME $u_1, \dots, u_N \in S$
$a_1, \dots, a_N \in \mathbb{F}$

NOTE BY CONVENTION, $\text{SPAN}(\varnothing) = \{0\}$

EX IS $-5x^2 - 2x + 6$ IN $\text{SPAN}\{x^2+3x+7,\ 4x^2+5x+7\}$?

$$-5x^2-2x+6 \overset{?}{=} a_1(x^2+3x+7) + a_2(4x^2+5x+7)$$
$$= a_1x^2 + 3a_1x + 7a_1 + 4a_2x^2 + 5a_2x + 7a_2$$
$$-5x^2 - 2x + 6 = (a_1+4a_2)x^2 + (3a_1+5a_2)x + 7a_1 + 7a_2$$

$$\begin{cases} a_1 + 4a_2 = -5 \\ 3a_1 + 5a_2 = -2 \\ 7a_1 + 7a_2 = 6 \end{cases} \Rightarrow \dots \Rightarrow \text{NO SOLUTION!} \quad \text{SO NO}$$

NOTE THINK OF SPAN AS THE INFO EXPRESSED BY A SET.
HERE, CANNOT EXPRESS $-5x^2-2x+6$ USING THE INFO WE HAVE, NEED ANOTHER PIECE OF INFO (OUT OF REACH)

EX WHAT IS SPAN $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$?

$$a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$$

ANS SET OF SYMMETRIC 2×2 MATRICES

NOW YOU MAY WONDER: HOW DOES THIS TIE BACK TO OUR CONCEPT OF SUBSPACE

$\Rightarrow$ III - SPAN IS A SUBSPACE

THEOREM IF S IS ANY SUBSET OF V (VS), THEN SPAN(S) IS A SUBSPACE OF V

THIS GIVES US YET ANOTHER WAY OF PRODUCING SUBSPACES: TAKE ANY SET AND CONSIDER ITS SPAN

NOTE IF $S = \emptyset$, THEN SPAN(S) = SPAN$(\emptyset) = \{0\} \longrightarrow$ SUBSPACE

SO ASSUME $S \neq \emptyset$

PROOF CHECK (a) - (c) HOLD FROM LAST TIME

(a) $0 \in$ SPAN(S) ?

LET x BE ANY ELEMENT OF $S \neq \emptyset$, THEN $0 = 0x \in$ SPAN(S) ✓

(b) IF x & y ARE IN SPAN(S), THEN:

$x = a_1 u_1 + \cdots + a_n u_n$, $y = b_1 v_1 + \cdots + b_m v_m$, $a_i, b_j \in \mathbb{F}$

$u_i, v_j \in S$

THEN $x + y = a_1 u_1 + \cdots + a_n u_n + b_1 v_1 + \cdots + b_m v_m \in$ SPAN(S)

(B/C LINEAR COMBO OF ELEMENTS IN S)

SO $x + y \in$ SPAN(S)

(c) IF $x \in \text{SPAN}(S)$ AND $c \in \mathbb{F}$, THEN

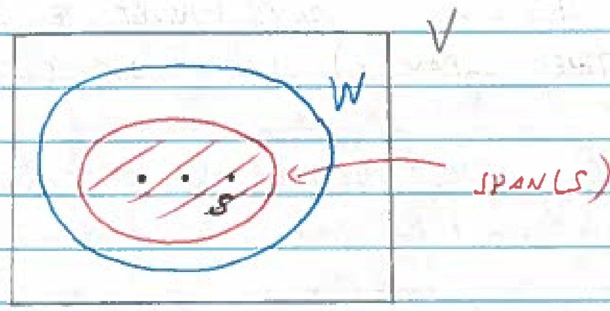
$$x = a_1 u_1 + \cdots + a_n u_n, \quad a_1, \ldots, a_n \in \mathbb{F}, \quad u_1, \ldots, u_n \in S$$

THEN $cx = c(a_1 u_1 + \cdots + a_n u_n) = (ca_1)u_1 + \cdots + (ca_n)u_n \in \text{SPAN}(S)$ ✓

HENCE $\text{SPAN}(S)$ IS A SUBSPACE OF V ∎

BUT WAIT, THERE'S MORE! $\text{SPAN}(S)$ IS NOT ONLY A SUBSPACE, BUT AN "OPTIMAL" SUBSPACE IN THE FOLLOWING SENSE

PICTURE



CLAIM IF W IS A SUBSPACE OF V AND $S \subseteq W$, THEN $\text{SPAN}(S) \subseteq W$ AS WELL

SAYS: ANY SUBSPACE OF V CONTAINING S MUST ALSO CONTAIN $\text{SPAN}(S)$

IN OTHER WORDS: $\text{SPAN}(S)$ IS THE SMALLEST SUBSPACE CONTAINING S ("OPTIMAL SPACE" CONTAINING S, ANY OTHER SPACE IS STRICTLY BIGGER)

WHY? LET W BE A SUBSPACE OF V WITH $S \subseteq W$
SHOW $\text{SPAN}(S) \subseteq W$

LET $x \in \text{SPAN}(S)$

THEN $x = a_1 u_1 + \cdots + a_n u_n$, $u_1, \ldots, u_n \in S$, $a_1, \ldots, a_n \in \mathbb{F}$

BUT SINCE $u_1, \dots, u_n \in S$ AND $S \subseteq W$, WE HAVE $u_1, \dots, u_n \in W$

AND SINCE $u_1, \dots, u_n \in W$ AND W IS A SUBSPACE,

$x = a_1 u_1 + \cdots + a_n u_n \in W$ ✓ (SEE HW #2)

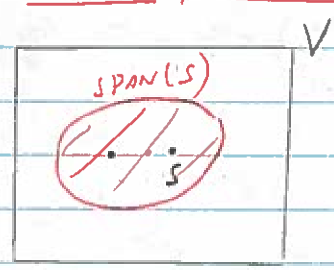
SO $x \in W$ AND HENCE $\text{SPAN}(S) \subseteq W$ ■

IV – SPANNING SETS

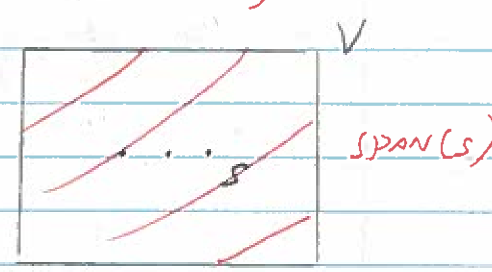
NOW ONE LAST QUESTION WE COULD ASK IS: HOW BIG IS THE SPAN OF A SET? COULD IT BE EQUAL TO THE WHOLE SPACE V?

DEF S SPANS / GENERATES V IF $\text{SPAN}(S) = V$

NON-EX



EX



EX LET $S = \{x^2, x^2+x, x^2+x+1\}$

CLAIM S SPANS P_2

WHY? LET $ax^2+bx+c \in P_2$ BE ARBITRARY

WTF a_1, a_2, a_3 WITH

$$\begin{aligned} ax^2+bx+c &= a_1 x^2 + a_2(x^2+x) + a_3(x^2+x+1) \\ &= a_1 x^2 + a_2 x^2 + a_2 x + a_3 x^2 + a_3 x + a_3 \\ &= \underline{(a_1+a_2+a_3)} x^2 + \underline{(a_2+a_3)} x + \underline{a_3} \\ &= \underline{a} \qquad\quad x^2 + \underline{b} \quad x + \underline{c} \end{aligned}$$

GIVEN

$$\Rightarrow \begin{cases} a_1 + a_2 + a_3 = a \\ a_2 + a_3 = b \\ a_3 = c \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = a - a_2 - a_3 = a - (b-c) - c = a - b \\ a_2 = b - a_3 = b - c \\ a_3 = c \end{cases}$$

SO $\quad ax^2 + bx + c = (a-b)x^2 + (b-c)(x^2+x) + c(x^2+x+1)$

NOW OF COURSE THIS IS A HORRIBLY INEFFICIENT WAY TO DETERMINE IF A SET GENERATES THE WHOLE SPACE OR NOT. LUCKILY, THERE ARE SOME WAY MORE EFFICIENT WAYS, SO STICK AROUND UNTIL NEXT TIME!

APPENDIX HOW TO SOLVE

$$\begin{cases} a_1 + 2a_2 + a_3 = 0 \\ a_1 - a_2 + 3a_3 = -2 \\ 3a_1 \quad\quad - a_3 = 4 \end{cases}$$

AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -1 & 3 & -2 \\ 3 & 0 & -1 & 4 \end{array}\right] \quad (x-1) \quad (x-3)$$

(EROS: INTERCHANGE OR MULTIPLY ROW OR ADD ROW TO ANOTHER)

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 2 & -2 \\ 0 & -6 & -4 & 4 \end{array}\right] \quad (x-2)$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & -8 & 8 \end{array}\right] \quad (\div -8)$$

ROW-ECHELON FORM (REF)
(THINGS TO THE LEFT OF THE "PIVOTS" ARE 0)

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad (x-2) \quad (x-1)$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad (\div -3)$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad (x-2) \quad \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

REDUCED REF (RREF)
(PIVOTS = 1, ANYTHING ABOVE PIVOT = 0)

$$\Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 0 \\ a_3 = -1 \end{cases}$$