LECTURE 3 - LINEAR COMBINATIONS AND SPAN (SECTION 1.4)

FRIDAY, APRIL 5, 2019

HAPPY FRIDAY, AND TUESDAY! WE’LL DISCOVER A NEAT WAY OF COMBINING VECTORS, WHICH NATURALLY LEADS TO THE CONCEPT OF A LINEAR COMBINATION.

I - LINEAR COMBINATIONS

DEFF. \( x \in V \) IS A LINEAR COMBINATION OF \( U_1, \ldots, U_n \in V \) IF

\[ x = a_1 U_1 + \cdots + a_n U_n \quad \text{for some} \ a_1, \ldots, a_n \in \mathbb{F} \]

**EX.** \( \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \) IS A LINEAR COMBINATION OF \( U_1 = \begin{pmatrix} 10 \\ 01 \end{pmatrix} \) AND \( U_2 = \begin{pmatrix} 01 \\ 10 \end{pmatrix} \)

[\( \text{Hence} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = 2 \begin{pmatrix} 10 \\ 01 \end{pmatrix} + 3 \begin{pmatrix} 01 \\ 10 \end{pmatrix} \)]

\[ X = a_1 U_1 + a_2 U_2 \]

**EX.** IS \( X = (0, -2, 4) \) A LINEAR COMBINATION OF \( U_1 = (1, 1, 3), U_2 = (2, -1, 0), U_3 = (1, 3, -1) \)?

AND THERE \( a_1, a_2, a_3 \) WITH \( U_3 = (a_1 U_1 + a_2 U_2 + a_3 U_3) \)?

\[ X = a_1 U_1 + a_2 U_2 + a_3 U_3 \]

\[ (0, -2, 4) = a_1 (1, 1, 3) + a_2 (2, -1, 0) + a_3 (1, 3, -1) \]

\[ (0, -2, 4) = (a_1 + 2a_2 + a_3, a_1 - a_2 + 3a_3, 3a_1 - a_3) \]

\[ \begin{cases} a_1 + 2a_2 + a_3 = 0 \\ a_1 - a_2 + 3a_3 = -2 \\ 3a_1 - a_3 = 4 \end{cases} \]

SYSTEM OF EQUATIONS

SOLVE THIS (USING ROW-REDUCTION - SEE PAGE 7, OR SEE TECHNIQUE IN THE BOOK)
\[
\begin{align*}
\begin{aligned}
& a_1 = 1 \\
& a_2 = 0 \\
& a_3 = -1
\end{aligned}
\end{align*}
\]

\[
(0, -2, 4) = 1(1, 1, 3) + 0(2, -1, 0) + (-1)(1, 3, -1)
\]

Now, of course, once you've taken one linear combo, you may ask: What about all the possible linear combinations?

And this is indeed useful and has its own name.

**II - Span**

**Def:** If \( S \) is any subset of \( V \), then \( \text{span}(S) \) is the set of all finite linear combinations of vectors in \( V \).

\[
x \in \text{span}(S) \iff x = a_1v_1 + \cdots + a_nv_n \text{ for some } v_1, \ldots, v_n \in S
\]

**Note:** By convention, \( \text{span}(\emptyset) = \{0\} \).

**Ex:** Is \( -5x^2 - 2x + 6 \) in \( \text{span}\{x^2 + 3x + 7, 4x^2 + 5x + 7\} \)?

\[
\begin{align*}
-5x^2 - 2x + 6 &= a_1(x^2 + 3x + 7) + a_2(4x^2 + 5x + 7) \\
&= a_1x^2 + 3a_1x + 7a_1 + 4a_2x^2 + 5a_2x + 7a_2 \\
-5x^2 - 2x + 6 &= (a_1 + 4a_2)x^2 + (3a_1 + 5a_2)x + 7a_1 + 7a_2
\end{align*}
\]

\[
\begin{align*}
a_1 + 4a_2 &= -5 \\
3a_1 + 5a_2 &= -2 \\
7a_1 + 7a_2 &= 6
\end{align*}
\]

\[\text{No solution! So } \boxed{\text{No}}\]

**Note:** Think of span as the info expressed by a set. Here, cannot express \( -5x^2 - 2x + 6 \) using the info we have; need another piece of info (out of reach).
EX \[ \text{WHAT IS } \text{SPAN} \left\{ \begin{bmatrix} 10 \\ 00 \end{bmatrix}, \begin{bmatrix} 01 \\ 10 \end{bmatrix}, \begin{bmatrix} 00 \\ 01 \end{bmatrix} \right\} \]

\[ = a_1 \begin{bmatrix} 10 \\ 00 \end{bmatrix} + a_2 \begin{bmatrix} 01 \\ 10 \end{bmatrix} + a_3 \begin{bmatrix} 00 \\ 01 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \]

Any set of symmetric 2x2 matrices

Now you may wonder: How does this tie back to our concept of subspace?

\( \therefore \text{SPAN IS A SUBSPACE} \)

**Theorem** \[ \text{IF } S \text{ IS ANY SUBSET OF } V \ (VS), \]
\[ \text{THEN } \text{SPAN}(S) \text{ IS A SUBSPACE OF } V \]

This gives us yet another way of producing subspaces: Take any set \( S \) and consider its span.

**Note** \[ \text{IF } S = \emptyset, \text{ THEN } \text{SPAN}(S) = \text{SPAN}(\emptyset) = \{0\} \rightarrow \text{SUBSPACE} \]

**Proof** \[ \text{CHECK (a)-(c) HOLD FROM LAST TIME} \]

(a) \[ 0 \in \text{SPAN}(S) \]

(b) \[ \text{IF } x, y \in \text{SPAN}(S), \text{ THEN:} \]
\[ x = a_1 u_1 + \cdots + a_n u_n, \quad y = b_1 v_1 + \cdots + b_m v_m, \quad a_i, b_j \in \mathbb{F} \]
\[ u_i, v_j \in S \]

Then \[ x + y = a_1 u_1 + \cdots + a_n u_n + b_1 v_1 + \cdots + b_m v_m \in \text{SPAN}(S) \]

(c) \[ \text{LINEAR COMB OF ELEMENTS IN } S \]
\[ \text{SO } x + y \in \text{SPAN}(S) \]
(c) If \( x \in \text{span}(S) \) and \( c \in F \), then

\[
x = a_1 u_1 + \cdots + a_n u_n, \quad a_1, \ldots, a_n \in F, \quad u_1, \ldots, u_n \in S
\]

then \( cx = c(a_1 u_1 + \cdots + a_n u_n) = (ca_1)u_1 + \cdots + (ca_n)u_n \in \text{span}(S) \).

Hence \( \text{span}(S) \) is a subspace of \( V \).

But wait, there's more! \( \text{span}(S) \) is not only a subspace, but an "optimal" subspace in the following sense.

**Claim**: If \( W \) is a subspace of \( V \) and \( S \subseteq W \), then \( \text{span}(S) \subseteq W \) as well.

**Points**: Any subspace of \( V \) containing \( S \) must also contain \( \text{span}(S) \).

In other words: \( \text{span}(S) \) is the smallest subspace containing \( S \) ("optimal" in the sense that any other subspace is strictly larger).

**Why?** Let \( W \) be a subspace of \( V \) with \( S \subseteq W \).

Show \( \text{span}(S) \subseteq W \).

Let \( x \in \text{span}(S) \).

Then \( x = a_1 u_1 + \cdots + a_n u_n, \quad u_1, \ldots, u_n \in S, \quad a_1, \ldots, a_n \in F \).
Since $u_1, \ldots, u_n \in S$ and $S \subseteq W$, we have $u_1, \ldots, u_n \in W$.

And since $u_1, \ldots, u_n \in W$ and $W$ is a subspace, $x = a_1 u_1 + \cdots + a_n u_n \in W \checkmark$ (see HW #2).

So $x \in W$ and hence spans $S \subseteq W$.

Now one last question we could ask is: How big is the span of a set? Could it be equal to the whole space $V$?

**Def.** $S$ spans/generates $V$ if $\text{span}(S) = V$.

Non-EX

<table>
<thead>
<tr>
<th>$\text{span}(S)$</th>
<th>$V$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-EX</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Ex.** Let $S = \{x^2, x^2 + x, x^2 + x + 1\}$

**Claim.** $S$ spans $P_2$.

**Why?** Let $a x^2 + b x + c \in P_2$ be arbitrary.

**WIF** $a, b, c$ are with

$$a x^2 + b x + c = a_1 x^2 + a_2 (x^2 + x) + a_3 (x^2 + x + 1)$$
$$= a_1 x^2 + a_2 x^2 + a_2 x + a_3 x^2 + a_3 x + a_3$$
$$= (a_1 + a_2 + a_3) x^2 + (a_2 + a_3) x + a_3$$
$$= a_1 x^2 + b x + c$$
\[ \begin{align*}
\Rightarrow & \quad \begin{cases} a_1 + a_2 + a_3 = a \\ a_2 + a_1 = b \\ a_1 = c \end{cases} \\
\Rightarrow & \quad \begin{cases} a_1 = a - a_2 - a_3 = a - (b - c) - c = a - b \\ a_2 = b - a_1 = b - c \\ a_3 = c \end{cases} \\
\end{align*} \]

so \[ a x^3 + b x + c = (a - b) x^2 + (b - c) (x^2 + x) + c (x^2 + x + 1) \]

Now of course this is a horribly inefficient way to determine if a set generates the whole space or not. Luckily, there are some ways more efficient ways, so stick around until next time!
APPENDIX: How to solve

\[
\begin{align*}
  a_1 + 2a_2 + a_3 &= 0 \\
  a_1 - a_2 + 3a_3 &= -2 \\
  3a_1 - a_3 &= 4
\end{align*}
\]

AUGMENTED MATRIX

\[
\begin{bmatrix}
  1 & 2 & 1 & 0 \\
  0 & -1 & 3 & -2 \\
  3 & 0 & -1 & 4
\end{bmatrix}
\]

\((x-1)(x-3)\)

(Enos: INTERCHANGE on MUARP in NOW or ANY NOW TO AUGMENT)

\[
\begin{bmatrix}
  1 & 2 & 1 & 0 \\
  0 & -1 & 3 & -2 \\
  0 & -6 & -4 & 4
\end{bmatrix}
\]

\((x-2)\)

\[
\begin{bmatrix}
  1 & 2 & 1 & 0 \\
  0 & -1 & 3 & -2 \\
  0 & 0 & -8 & 4
\end{bmatrix}
\]

\((\div -8)\)

NOW-ECHELON FORM

\((\text{NEF})\)

(THINGS TO THE LEFT OF THE PIVOTS AN E 0)

\[
\begin{bmatrix}
  1 & 2 & 1 & 0 \\
  0 & -1 & 3 & -2 \\
  0 & 0 & 1 & 1
\end{bmatrix}
\]

\((x-1)\)

\[
\begin{bmatrix}
  1 & 2 & 0 & 1 \\
  0 & -3 & 0 & 0 \\
  0 & 0 & 1 & -1
\end{bmatrix}
\]

\((\div -3)\)

\[
\begin{bmatrix}
  1 & 2 & 0 & 1 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & -1
\end{bmatrix}
\]

\((x-2)\)

\[
\begin{bmatrix}
  1 & 0 & 0 & 1 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & -1
\end{bmatrix}
\]

\[
\begin{align*}
  a_1 &= 1 \\
  a_2 &= 0 \\
  a_3 &= -1
\end{align*}
\]

REDUCED NEF \((\text{RNEF})\)

(PIVOTS = \(1\), ANYTHING ABOVE PIVOT = 0)