MONDAY, APRIL 8, 2019

LECTURE 4 – LINEAR DEPENDENCE AND INDEPENDENCE (SECTION 1.5)

TODAY WE'RE GOING TO DISCUSS THE OTHER SIDE OF SPAN, WHICH IS LINEAR DEPENDENCE AND INDEPENDENCE

I – LINEAR DEPENDENCE

(INTUITIVELY: SAYS VECTORS ARE RELATED)

DEF A SUBSET $S \subseteq V$ IS LINEARLY DEPENDENT (LD) IF THERE ARE VECTORS $v_1, \dots, v_n$ IN S AND $a_1, \dots, a_n \in \mathbb{F}$ NOT ALL 0 SUCH THAT

$$a_1 v_1 + \cdots + a_n v_n = \underline{0}$$

EX $S = \{(1,0), (0,1), (0,2)\}$ IS LD B/C

$$\underline{\underline{0}}\,(1,0) + \underline{\underline{2}}\,(0,1) + \underline{\underline{(-1)}}(0,2) = (0,0)$$

REMARKS
1) SOME OF THE a_i COULD BE 0, JUST NOT ALL OF THEM
2) WE ALWAYS HAVE $0 v_1 + \cdots + 0 v_n = \underline{0}$, NEED NONTRIVIAL LINEAR COMBO
3) S LD $\not\Rightarrow$ v_1 IS A LINEAR COMBO OF $v_2, \dots, v_n$!
(THAT'S WHY WE HAVE SUCH A WEIRD DEF)
4) $\varnothing$ IS LIN IND

EX $S = \{\cos^2(x), \sin^2(x), 1\}$ LD B/C

$$1 \cos^2(x) + 1 \sin^2(x) + (-1)\, 1 = 0$$

II - LINEAR INDEPENDENCE

ON THE OTHER SIDE OF THE SPECTRUM, THERE'S THE CONCEPT OF LI, WHICH IS JUST THE NEGATION OF LD (NEGATION, PLEASE)

DEF S IS LINEARLY INDEPENDENT (LI) IF FOR ALL $U_1, \dots, U_N \in S$ AND FOR ALL $a_1, \dots, a_N \in \mathbb{F}$

$$a_1 U_1 + \cdots + a_N U_N = 0 \Rightarrow a_1 = 0, \dots, a_N = 0$$

(THE ONLY LIN COMB THAT GIVES YOU $\underline{0}$ IS THE TRIVIAL LINEAR COMBO)

EX IS $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ LI ?

SUPPOSE $a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a_1 + a_2 + a_3 + a_4 & a_2 + a_3 + a_4 \\ a_3 + a_4 & a_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a_1 + a_2 + a_3 + a_4 = 0 \\ a_2 + a_3 + a_4 = 0 \\ a_3 + a_4 = 0 \\ a_4 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \\ a_4 = 0 \end{cases}$$

YES

IN GENERAL SOLVE A SYSTEM OF EQUATIONS

↳ IF $a_1 = 0, \dots, a_N = 0$ ⇝ LI

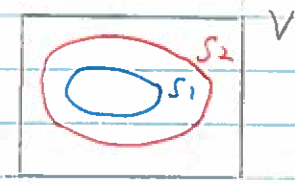
↳ IF NONZERO SOLUTION ⇝ LD

III – Some properties (to practice w/ the def of L.D.)

EX CLAIM Any set S with $\underline{0}$ is lin dep

WHY? Let $u_1 = \underline{0}$, $a_1 = 1$, then $a_1 \underline{u_1} = \underline{0}$ but $a_1 \neq 0$

EX CLAIM Suppose $S_1 \subseteq S_2 \subseteq V$



(a) If S_1 is LD, then S_2 is LD

WHY? Since S_1 is LD, there are $u_1, \dots, u_N \in S_1$, $a_1, \dots, a_N$ not all 0 with

$$a_1 u_1 + \cdots + a_N u_N = \underline{0}$$

But since $S_1 \subseteq S_2$, $u_1, \dots, u_N \in S_2$,

so there are $v_1, \dots, v_N \in S_2$ and $b_1, \dots, b_N$ not all 0 with

$$b_1 v_1 + \cdots + b_N v_N = \underline{0}$$

Namely $v_1 = u_1, \dots, v_N = u_N$, $b_1 = a_1, \dots, b_N = a_N$ (not all 0)

So S_2 is LD

(b) If S_2 is LI then S_1 is LI

Contrapositive of (a)!

IV – LINEAR DEPENDENCE AND SPAN

NOTE LI SETS ARE NICE, THEN SPAN IS WHAT YOU THINK IT IS!

EX $\text{SPAN}\underbrace{\{(1,0,1),(1,1,0)\}}_{LI}$ IS A PLANE IN $\mathbb{R}^3$

EX $\text{SPAN}\underbrace{\{(1,0,1),(2,0,2)\}}_{LD}$ IS NOT A PLANE IN $\mathbb{R}^3$

IN FACT IF S IS LD, CAN REMOVE VECTORS FROM S WITHOUT CHANGING THE SPAN OF S (MATH JENGA)

EX $S = \{(1,0,0),(0,1,0),\cancel{(2,3,0)}\}$ LD

NOTICE $(2,3,0)$ IS A LINEAR COMBO OF $(1,0,0)$, $(0,1,0)$, SO IF

$S' = \{(1,0,0),(0,1,0)\}$ (S BUT W/ $(2,3,0)$ REMOVED)

THEN $\text{SPAN}(S') = \text{SPAN}(S)$ (AND IN FACT, CAN SUCCESSIVELY REMOVE REDUNDANT VECTORS THAT WAY)

BUT IF S IS LI, CANNOT REMOVE VECTORS W/O CHANGING THE SPAN, SO LI SETS ARE OPTIMAL IN THIS SENSE

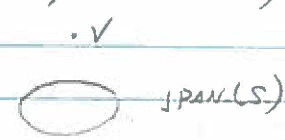
WHAT ABOUT ADDING VECTORS? CAN YOU ADD VECTORS TO A LI SET AND STILL GET A LI SET? THIS IS THE POINT OF THE NEXT THEOREM, WHICH WILL BE USED NEXT TIME

V – THE "INTRUDER" THEOREM

MOTIVATION

EX LET $S = \{(1,0,0), (0,1,0)\}$ LI , $v = (0,0,1)$

NOTICE $v \notin span(S)$



THEN $S \cup \{v\} = \{(1,0,0), (0,1,0), (0,0,1)\}$ IS STILL LI!

AND THIS IS ALWAYS TRUE: AS LONG AS YOU ADD VECTORS THAT ARE NOT IN YOUR SPAN, YOUR NEW SET REMAINS LI. AND THIS IS THE POINT OF WHAT I'D LIKE TO CALL:

THEOREM ["INTRUDER" THEOREM]

SUPPOSE $S \subseteq V$ IS LI AND $v \in V$, THEN

$$S \cup \{v\} \text{ LD} \Leftrightarrow v \in span(S)$$

(EQUIVALENTLY $v \notin span(S) \Leftrightarrow S \cup \{v\}$ LI (SEE ABOVE))

(WHY INTRUDER? SUPPOSE S IS LI = GOOD, BUT $S \cup \{v\}$ LD = BAD, THEN IT'S v's FAULT)

PROOF ($\Leftarrow$) IF $v \in span(S)$, THEN

$$v = a_1 u_1 + \cdots + a_N u_N \quad \text{FOR SOME } u_1, \ldots, u_N \in S, \; a_1, \ldots, a_N \in F$$

BUT THEN $1v - a_1 u_1 \cdots - a_N u_N = 0$

SINCE $v, u_1, \ldots, u_N \in S \cup \{v\}$ AND $1, -a_1, \ldots, -a_N$ NOT ALL 0,

$S \cup \{V\}$ IS LD ✓

$(\Rightarrow)$ SUPPOSE $S \cup \{V\}$ LD (SHOW $V \in \text{SPAN}(S)$)

THEN THERE ARE $U_1, \dots, U_N \in S \cup \{V\}$, $a_1, \dots, a_N$ NOT ALL 0 WITH

$$a_1 U_1 + \cdots + a_N U_N = \underline{0} \quad (*)$$

IF ALL THE $U_1, \dots, U_N$ ARE IN S, THEN $(*)$ IMPLIES S IS LD $\Rightarrow\Leftarrow$

SO ONE OF $U_1, \dots, U_N$ MUST BE EQUAL TO V, SAY $U_1 = V$

THEN $(*)$ SAYS $\quad a_1 V + a_2 U_2 + \cdots + a_N U_N = \underline{0} \quad (**)$

THEN $\quad a_1 V = -a_2 U_2 \cdots - a_N U_N$

$\Rightarrow \quad V = -\dfrac{a_2}{a_1} U_2 \cdots - \dfrac{a_N}{a_1} U_N \in \text{SPAN}(S)$ PROVIDED $a_1 \neq 0$

BUT IF $a_1 = 0$, THEN $(**)$ SAYS $a_2 U_2 + \cdots + a_N U_N = 0$
SO $a_1 = 0$ AND $a_2 = 0, \dots, a_N = 0$ (SINCE S IS LI)
BUT THIS CONTRADICTS $(*)$ ($S \cup \{V\}$ LD) ■