WEDNESDAY, APRIL 10, 2019

LECTURE 5 – BASIS AND DIMENSION (I) (SECTION 1.6)

WELCOME TO THE HEART AND SOUL OF LINEAR ALGEBRA: BASIS AND DIMENSION. TODAY WE'LL SHOW THAT EVEN THOUGH VS ARE INFINITE, WE CAN DESCRIBE THEM USING ONLY FINITELY MANY VECTORS.

I – DEFINITION AND EXAMPLES

DEF
1) A **BASIS** β FOR V IS A LI SET WITH $\mathrm{SPAN}(\beta) = V$
2) $\mathrm{DIM}(V)$ = NUMBER OF VECTORS IN ANY BASIS FOR V

(SO TO SHOW β IS A BASIS, JUST SHOW IT'S LI AND THAT IT SPANS V, TO FIND $\mathrm{DIM}(V)$, JUST COUNT THE # OF VECTORS IN V)

EX 1 $V = \mathbb{R}^N$

1) BASIS $\beta = \{e_1, \dots, e_N\}$ $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, \dots, 0)$, … $e_N = (0, \dots, 0, 1)$ LI ✓ SPANS ✓

2) $\mathrm{DIM}(\mathbb{R}^N) = N$

EX 2 $V = P_N$ (POLYS OF DEG $\leq N$)

1) BASIS $\beta = \{1, x, \dots, x^N\}$

2) $\mathrm{DIM}(P_N) = N+1$ ⚠

EX 3 $V = P$ (ALL POLYS, x^2+1, x^3-2, …)

1) BASIS $\beta = \{1, x, x^2, x^3, \dots\}$

2) $\mathrm{DIM}(P) = \infty$ (SOME VS ARE ∞-DIMENSIONAL)

EX 4 $V = \{0\}$

1) BASIS $\beta = \emptyset$

2) $\dim\{0\} = 0$

TWO ISSUES 1) DOES EVERY VS HAVE A BASIS?
2) DO ANY 2 BASES OF V HAVE THE SAME NUMBER OF VECTORS? (O/W DIM ISN'T WELL-DEFINED!)

II – EXISTENCE OF A BASIS

FACT EVERY VS HAS A BASIS ~> SEE SECTION 1.7 (USES ZORN'S LEMMA)

(I'M NOT GOING TO SHOW THIS, BUT WHAT I'M GOING TO SHOW YOU IS A SPECIAL CASE, NAMELY EVERY FINITE-DIMENSIONAL VS HAS A BASIS)

THEOREM IF $V = \text{SPAN}(S)$ FOR SOME FINITE SET S, THEN SOME SUBSET OF S IS A BASIS FOR V (AND HENCE V HAS A BASIS)

EX SUPPOSE $V = \text{span}\{(2,-3,5), \cancel{(8,-12,20)}, (1,0,-2), (0,2,-1), \cancel{(7,2,0)}\}$

FIND A BASIS FOR V

IDEA START W/ $(2,-3,5)$ AND SUCCESSIVELY REMOVE LD VECTORS

1) $(2,-3,5)$ ✓
2) $(8,-12,20) \in \text{span}\{(2,-3,5)\}$ REMOVE
3) $(1,0,-2) \notin \text{span}\{(2,-3,5)\}$ KEEP
4) $(0,2,-1) \notin \text{span}\{(2,-3,5), (1,0,-2)\}$ KEEP
5) $(7,2,0) \in \text{span}\{(2,-3,5), (1,0,-2), (0,2,-1)\}$ REMOVE

ANS $\{(2,-3,5),(1,0,-2),(0,2,-1)\}$ (LI BY "INTRUDER" THEOREM, SPANS B/C WE REMOVED REDUNDANT VECTORS)

PROOF OF THEOREM

(NOTICE HOW WE HAVE DONE THINGS ITERATIVELY IN THE PREVIOUS EX)

INDUCTION ON N = SIZE OF $S = |S|$

1) LET P_N BE THE PROP " IF $V = \text{SPAN}(S)$, $|S| = N$, THEN SOME SUBSET OF S IS A BASIS FOR V "

2) BASE CASE $N=0$ $|S| = 0 \Rightarrow S = \emptyset$

THEN $V = \text{SPAN}(S) = \text{SPAN}(\emptyset) = \{0\}$

BASIS $= \emptyset \subseteq S$ ✓

3) INDUCTIVE STEP SUPPOSE P_N IS TRUE, SHOW P_{N+1} IS TRUE

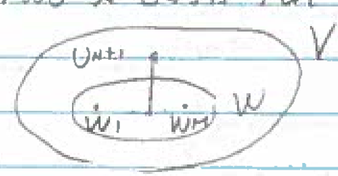
SUPPOSE $V = \text{SPAN}(S)$, $|S| = N+1$, SHOW SOME SUBSET OF S IS A BASIS

LET $S = \{u_1, \dots, u_{N+1}\}$

TRICK CONSIDER $S' = \{u_1, \dots, u_N\}$, $W = \text{SPAN}(S')$

BY THE INDUCTIVE HYPOTHESIS, SOME SUBSET OF S', SAY $\{w_1, \dots, w_m\}$ IS A BASIS FOR W (USE THIS TO FIND A BASIS FOR V)

CASE 1 $u_{N+1} \notin W$



THEN LET $\beta = \{\underbrace{w_1 \dots w_m}_{\subseteq S'}, u_{N+1}\} \subseteq S$ LI ✓ (BY INTRUDER THM) SPANS V ✓ (CHECK)

SO β IS A BASIS FOR V ✓

4

CASE 2 $\quad u_{n+1} \in W$ $\quad$ $w_1 \cdots w_m \; u_{n+1}$ W

LET $\beta = \{w_1, \ldots, w_m\} \subseteq S$ $\quad$ LI ✓

AND $\mathrm{span}(\beta) = \mathrm{span}\{w_1, \ldots, w_m\} = \mathrm{span}\{u_1, \ldots, u_n\} = \mathrm{span}\{u_1, \ldots, u_{n+1}\}$
$= V$ ✓

SO β IS A BASIS

HENCE P_{n+1} IS TRUE, SO P_n IS TRUE FOR ALL n ■

(SO FOR OUR PURPOSES, THE QUESTION OF EXISTENCE OF A BASIS IS SETTLED, AND NOW LET'S MOVE ON TO THE SECOND ISSUE: DO ANY 2 BASES OF V HAVE THE SAME NUMBER OF VECTORS? AND THIS FOLLOWS FROM THE FOLLOWING ULTRA IMPORTANT THEOREM, WITHOUT WHICH LA WOULDN'T EXIST)

III — THE REPLACEMENT THEOREM

MOTIVATION

EX LET $\quad LIN = \{(2,-3,5), (1,0,-2)\}$ $\quad$ LI, $m = 2$

$GEN = \{(8,-12,20), (7,2,0), (0,2,-1)\}$ GENERATES/SPANS $\mathbb{R}^3$, $n = 3$

REPLACEMENT THEOREM (BELOW) SAYS:

1) $|LIN| \leq |GEN|$ $\quad$ ($m \leq n$, $2 \leq 3$; A LI SET CAN NEVER BE STRICTLY BIGGER THAN A SPANNING SET)

2) CAN ADD $n - m = 3 - 2 = 1$ ELEMENT FROM GEN TO LIN TO GET A SET THAT ALSO SPANS $\mathbb{R}^3$

IN FACT LET $H = \{(0,2,-1)\} \subseteq GEN$

THEN $LIN \cup H = \{ \underbrace{(2,-3,5), (1,0,-2)}_{= LIN}, \underbrace{(0,2,-1)}_{\subseteq GEN} \}$ SPANS $\mathbb{R}^3$

ULTRA IMPORTANT [REPLACEMENT THEOREM] ("EXTENSION" THEOREM)

LET LIN BE A LI SUBSET OF V WITH M VECTORS
AND GEN BE A SPANNING SUBSET OF V WITH N VECTORS

THEN 1) $M \leq N$

2) THERE IS A SUBSET H of GEN WITH $N-M$ VECTORS SUCH THAT $LIN \cup H$ SPANS V

PICTURE $LIN = \square\,\square\,\square$ $(M=3)$, $GEN = \bullet\,\bullet\,\bullet\,\bullet\,\bullet$ $(N=5)$, $H = \bullet\,\bullet$ $(N-M=2)$

$LIN \cup H = \underbrace{\overset{M}{\square\,\square\,\square}}_{LIN}\ \underbrace{\overset{N-M}{\bullet\,\bullet}}_{H}$ (SPANS V)

REMARKS
1) REPLACEMENT B/C YOU'RE REPLACING VECTORS IN GEN W/ VECTORS IN LIN (USUALLY BETTER THAN GEN)
2) IT'S THE NUMBER THAT MATTERS, O/W YOU CAN ALWAYS JUST ADD ALL OF GEN TO LIN
3) $N-M$ B/C WE WANT $LIN \cup H$ TO HAVE SIZE (AT MOST $N = |GEN|$)

IV – CONSEQUENCES (THIS THEOREM IS IMPORTANT B/C OF ITS CONSEQUENCES)

COROLLARY 1 (SUPPOSE V HAS A FINITE BASIS)
EVERY BASIS OF V HAS THE SAME # OF VECTORS

WHY? LET β BE THAT FINITE BASIS WITH N VECTORS
LET γ BE ANOTHER BASIS OF V (POSSIBLY ∞)
INTUITIVELY: β LI, γ SPANS $\Rightarrow |\beta| \leq |\gamma|$, γ LI, β SPANS $\Rightarrow |\gamma| \leq |\beta|$

γ

$\underbrace{\cdot \; \cdot \; \overset{N+1}{\cdot} \; \cdot}_{LN} \; \cdot \; \cdot \; \cdot$

1) SUPPOSE $|\gamma| > |\beta|$, SO $|\gamma| > N$

<u>TRICK</u> LET $GEN = \beta$

$LN =$ ALL SUBSET OF γ WITH $N+1$ ELEMENTS ($|LN| = N+1$)

REPLACEMENT SAYS $|LN| \leq |GEN|$

$N+1 \leq N \Rightarrow \Leftarrow$

SO $|\gamma| \leq |\beta| = N$ (SO γ IS FINITE)

2) NOW LET $LN = \beta$, $GEN = \gamma$

REPLACEMENT: $|LN| \leq |GEN| \Rightarrow |\beta| \leq |\gamma|$

3) HENCE $|\beta| = |\gamma|$ ∎