LECTURE 5 - BASIS AND DIMENSION (I) (SECTION 1.6)

WELCOME TO THE HEART AND SOUL OF LINEAR ALGEBRA: BASIS AND DIMENSION. TODAY WE'LL SHOW THAT EVEN THOUGH VS ARE INFINITE, WE CAN DESCRIBE THEM USING ONLY FINITELY MANY VECTORS.

I - DEFINITION AND EXAMPLES

DEF: 1) A BASIS \( \beta \) for \( V \) is a set with \( \text{span}(\beta) = V \)

2) \( \dim(V) = \text{number of vectors in any basis for } V \)

(SO TO SHOW \( \beta \) IS A BASIS, JUST SHOW IT'S LI AND THAT IT SPANS \( V \).
TO FIND \( \dim(V) \), JUST COUNT THE # OF VECTORS IN \( \beta \).)

EX 1 \( V = \mathbb{R}^n \)

1) Basis \( \beta = \{ e_1, \ldots, e_n \} \)
   \( e_1 = (1, 0, \ldots, 0) \), \( e_2 = (0, 1, \ldots, 0) \), \ldots , \( e_n = (0, \ldots, 0, 1) \)

2) \( \dim(\mathbb{R}^n) = n \)

EX 2 \( V = P_n \) (POLYS OF DEG \( \leq n \))

1) basis \( \beta = \{ 1, x, \ldots, x^n \} \)

2) \( \dim(P_n) = n+1 \) \( \Delta \)

EX 3 \( V = P \) (ALL POLYS, \( x^4 + 1, x^3 - 1, \ldots \))

1) Basis \( \beta = \{ 1, x, x^2, x^3, \ldots \} \)

2) \( \dim(P) = \infty \) (SOME VS ARE \( \infty \)-DIMENSIONAL)
EX. \[ V = \{ v \} \]

1) \[ B \] = \[ \emptyset \]

2) \[ \dim \{ 0 \} = 0 \]

**Two Issues**

1) **Does Every Vector Space Have a Basis?**
2) **Do Any Two Bases of \( V \) Have the Same Number of Vectors?** (Note: \( \dim \) is not well-defined!)

**Existence of a Basis**

**Fact** Every Vector Space has a basis (see Section 1.7; uses Zorn’s Lemma).

(I’m not going to show this, but what I’m going to show you is a special case, namely every finite-dimensional vector space has a basis.)

**Theorem** If \( V = \text{span} \{ S \} \) for some finite set \( S \), then some subset of \( S \) is a basis for \( V \) (and hence \( V \) has a basis).

**Ex.** Suppose \( V = \text{span} \{ (2,-3,5), (8,-14,20), (1,0,-2), (0,2,-1) \} \). Find a basis for \( V \).

*Idea: Test \( (2,-3,5) \) and successively remove linearly dependent vectors.*

1) \( (2,-3,5) \)

2) \( (8,-14,20) \) is \( \text{span} \{ (2,-3,5) \} \) 
   \[ \xrightarrow{\text{ redundant}} \]

3) \( (1,0,-2) \) is \( \text{span} \{ (2,-3,5) \} \) 
   \[ \xrightarrow{\text{ redundant}} \]

4) \( (0,2,-1) \) is \( \text{span} \{ (2,-3,5), (1,0,-2) \} \) 
   \[ \xrightarrow{\text{ redundant}} \]

5) \( (7,8,10) \) is \( \text{span} \{ (2,-3,5), (1,0,-2), (0,2,-1) \} \) 
   \[ \xrightarrow{\text{ redundant}} \]
PROOF OF THEOREM

(Notice how we have done things iteratively in the previous ex)

Induction on \( N = \text{size of } S = |S| \)

1) Let \( P_n \) be the prop "if \( V = \text{span}(S), |S| = n \), then some subset of \( S \) is a basis for \( V \)"

2) Base Case \( N = 0 \) \(|S| = 0 \Rightarrow S = \emptyset \)
   Then \( V = \text{span}(S) = \text{span}(\emptyset) = \{0\} \)
   \[ \text{basis } = \emptyset \subseteq S \cup V \]

3) Inductive Step
   Suppose \( P_n \) is true, show \( P_{n+1} \) is true

   Suppose \( V = \text{span}(S), |S| = n+1 \), show some subset of \( S \) is a basis

   Let \( S = \{u_1, \ldots, u_{n+1}\} \)

   **Claim**: Consider \( S' = \{u_1, \ldots, u_n\} \), \( W = \text{span}(S') \)

   By the inductive hypothesis, some subset of \( S' \), say \( \{w_1, \ldots, w_m\} \), is a basis for \( W \) (use this to find a basis for \( V \))

   **Case I**: \( u_{n+1} \notin W \)

   Then let \( \beta = \{w_1, \ldots, w_m, u_{n+1}\} \subseteq S \subseteq S' \)
   \( \text{span } V \) (by inductive hypothesis)
   \( \subseteq S' \)

   So \( \beta \) is a basis for \( V \)

   \[ \checkmark \]
Case 2 \[ \text{Unit } \in W \]

Let \( \beta = \{ W_1, \ldots, W_n \} \subseteq S \)

And \( \text{span} (\beta) = \text{span} \{ W_1, \ldots, W_n \} = \text{span} \{ U_1, \ldots, U_n \} = \text{span} \{ U_1, \ldots, U_{n+1} \} = \text{span} \)

so \( \beta \) is a basis

Hence \( \text{span} \) is true, so \( \text{span} \) is true for all \( N \)

(for our purposes, the question of existence of a basis is settled, and now let's move on to the second issue: do any 2 bases of \( V \) have the same number of vectors? And thus follows from the following important theorem, which is that \( A \) would not exist)

III - THE REPLACEMENT THEOREM

**Motivation**

**EX** Let \( \text{lin} = \{ (2, -3, 5), (1, 0, -2) \} \) LT, \( M = 2 \)

\( \text{gen} = \{ (8, -12, 20), (7, 2, 0), (5, 2, -1) \} \) GENERATE / SPANN

\( \text{replacement theorem} \) (below) **NOTE**

1) \( |\text{lin}| \leq |\text{gen}| \) (M.N. 2 \( \leq 3 \); a lin set can never be strictly bigger than a span

2) Can add \( N - M = 3 - 2 = 1 \) element from \( \text{gen} \) to \( \text{lin} \) to get a set that also \( \text{spans} \)

In fact, let \( \mathcal{A} = \{(0, 2, -1)\} \subseteq \text{gen} \)
Then \( \text{lin} \cup H = \left\{ (2, -3, 5), (1, 0, -4), (0, 2, -1) \right\} \) span \( \mathbb{R}^3 \)

\[ \text{lin} \subset \text{gen} \]

**Ultra Important** [Replacement Theorem] (Extension Theorem)

Let \( \text{lin} \) be a LI subset of \( V \) with \( M \) vectors

And \( \text{gen} \) be a spanning subset of \( V \) with \( N \) vectors

Then

1) \( M \leq N \) of

2) There is a subset \( H \) \( \subset \text{gen} \) with \( N-M \) vectors

such that \( \text{lin} \cup H \) spans \( V \)

Picture

\[ \text{lin} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \text{gen} = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}, \quad H = \cdots \]

\( (M=3) \quad (N=5) \quad (N-M=2) \)

\[ \text{lin} \cup H = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \] span \( V \)

**Remark**

1) Replacement b/c you're replacing vectors in \( \text{gen} \), \( \text{w/} \) vectors in \( \text{lin} \) (usually better than \( \text{gen} \))

2) It's the number that matters, if \( M \) you can always wipe out all of \( \text{gen} \) to \( \text{lin} \)

3) \( N-M \) b/c we want \( \text{lin} \cup H \) to have size (at most \( N = \text{gen} \))

**IV - Consequences** (This theorem is important b/c of its consequences)

**Corollary** (Suppose \( V \) has a finite basis)

Every basis of \( V \) has the same \# of vectors

Why?

Let \( \beta \) be that finite basis with \( N \) vectors

Let \( \hat{\beta} \) be another basis of \( V \) (possibly \( \neq \beta \))

(Recall: \( \hat{\beta} \cup \beta \) span \( V \))
1) Suppose \(|X| > |\tilde{I}|\), so \(|X| > N\). Let \(p\) be a subset of \(X\) with \(n + 1\) elements where \(|L| = n + 1\).

\[L = \text{A LI subset of } X \text{ with } n + 1 \text{ elements} \]

Replacement shows \(|L| \leq |\text{GEN}|\).

\[N + 1 \leq N \implies 0 \leq 1\]

So, \(|X| \leq |\tilde{I}| = N\) (so \(X\) is finite).

2) Now let \(|X| = |\tilde{I}|\), \(\text{GEN} = \varnothing\).

Replacement: \(|L| \leq |\text{GEN}| = |\tilde{I}| < |X|\).

3) Hence, \(|\tilde{I}| = |X|\) .