LECCIÓN 6 - BASIS AND DIMENSION (II) (SECCIÓN 1.3)

Previously on "That is to say," we discovered the Replacement Theorem, which is at the very heart of our adventure. Today we'll focus on its proof and further consequences.

I - THE REPLACEMENT THEOREM

Theorem. Let \( \text{lin} \) be a linear subset of \( V \) with \( M \) vectors.

Let \( \text{gen} \) be a spanning subset of \( V \) with \( N \) vectors.

Then:
1) \( M \leq N \)
2) There is a subset \( H \) of \( \text{gen} \) with \( N - M \) vectors such that \( \text{lin} \cup H \) spans \( V \)

Picture: \( \text{lin} = \{ \ldots \} \) \( \text{gen} = \{ \ldots \} \) \( H = \{ \ldots \} \) \( \text{lin} \cup H = \{ \ldots \} \)

(Intuitively: any LI set can be extended to a spanning set)

Proof. Induction on \( M = |\text{lin}| \)

Base case \( M = 0 \) then \( \text{lin} = \emptyset \) and check \( H = \text{gen} \)

Inductive step. Suppose \( \forall m \leq M \text{ true}, \) show \( P_{m+1} \text{ true} \)

Let \( \text{lin} = \{ V_1, \ldots, V_m \} \) and \( \text{gen} \) be given, find \( H \).

(Idea: how \( \{ \ldots \} \text{ span } V \) \( H \)

Know \( \{ \ldots \} \text{ span } V \)

But \( \{ \ldots \} \text{ span } \{ \ldots \} = V \), so \( \in \text{span} \{ \ldots \} \)

(We're replacing \( \cdot \) with \( \circ \)
STEP 1
Consider $L' = \{ V_1, \ldots, V_M \} \subseteq \text{GEN as above.}

Then by ind. hyp.: 1) MEN

2) There is a subset $H' = \{ u_1, \ldots, u_{n-M} \} \subseteq \text{GEN}

such that $L' \cup H' = \{ V_1, \ldots, V_M, u_1, u_2, \ldots, u_{n-M} \}$ upon $V(\ast)$.

STEP 2
Consider $V_{MH} \in V$

By (\ast), there are $a_1, \ldots, a_M, b_1, \ldots, b_{n-M} \in F$ such that

$$V_{MH} = a_1 V_1 + \ldots + a_M V_M + b_1 U_1 + \ldots + b_{n-M} U_{n-M} \quad (\ast\ast)$$

Note: If $N=M$, then $H' = \emptyset$ so (\ast\ast) becomes

$$V_{MH} = a_1 V_1 + \ldots + a_M V_M$$

which contradicts $\text{lin}(V_1, \ldots, V_M) \subseteq \text{LT} \implies N > M \Rightarrow N \geq M+1 \Rightarrow 1)$ of P.M.\(+1\) \checkmark

STEP 3
Similarly, in (\ast\ast), $b_1, \ldots, b_{n-M}$ cannot be all 0, so one of them, say $b_1 \neq 0$

so $b_1 U_1 = -a_1 V_1 - \ldots - a_M V_M + V_{MH} - b_2 U_2 - \ldots - b_{n-M} U_{n-M}$

$$U_1 = \frac{-a_1}{b_1} V_1 - \ldots - \frac{a_M}{b_1} V_M + \frac{1}{b_1} V_{MH} - \frac{b_2}{b_1} U_2 - \ldots - \frac{b_{n-M}}{b_1} U_{n-M} \quad (\ast\ast\ast)$$

so $U_1 \in \text{span}(V_1, \ldots, V_M, V_{MH}, U_2, \ldots, U_{n-M}) \subseteq \text{span}(a_1 \ldots a_M U_1, \ldots, U_{n-M})$

Let $H = \{ u_1, \ldots, u_{n-M} \} \subseteq \{ u_1, \ldots, u_{n-M} \} = H' \subseteq \text{GEN}$

Then: 1) $H \cup H', N-M-1 = N-(M+1)$ elements
2) \( \text{LIN}(U) = \{ V_1, \ldots, V_{M+1} \} U \{ U_1, \ldots, U_{M+1} \} \) spans \( V \)

Because \( \{ V_1, \ldots, V_{M+1} \} U \) spans \( V \) (by (1))

And \( U \) spans \( \{ U_1, \ldots, U_{M+1} \} \) (check)

Hence \( P_{M+1} \) is true, so \( P_n \) is true for all \( M \).

**II - Importance of Dimension**

The reason the Replacement Theorem is important is not its statement, but its consequences. We've already seen one last time (any 2 bases of \( V \) have the same set of elements). Here's another one.

**Corollary:** Suppose \( \text{DIM}(V) = N \), then any \( L \) subset of \( V \) with \( N \) vectors is a basis for \( V \).

**EX:** \( \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \} \) is a L subset of \( V = \text{Max}^4 \) with 4 vectors, and \( \text{DIM}(V) = 2(2) = 4 \), so it is a basis (\( \Rightarrow \) spans \( V \))

Why? Let \( L \) be a L subset of \( V \) with \( N \) vectors.

Use Replacement with: \( \text{LIN} = L \) (\( N \) vectors)

\( \text{GEN} = \{ \} \) (any basis of \( V \) \( N \) vectors)

Then there is a subset \( H \) of \( \text{GEN} \) with \( N - N = 0 \) vectors (\( \Rightarrow H = \emptyset \)) such that \( \text{LIN}(U) \) spans \( V \)

so \( \text{LIN}(U) = L U \emptyset = L \) spans \( V \)

so \( L \) is LI and spans \( V \), so \( L \) is a basis of \( V \).
Claim 1) Any (finite) set that spans $V$ must have $\geq N$ vectors.

Claim 2) Any spanning set of $V$ with $N$ vectors is a basis for $V$.

Example: $\{x^5, x^4 + x, x^2 + x + 1\}$ spans $\mathbb{P}_2$ and $\dim(\mathbb{P}_2) = 3$, so it's a basis for $\mathbb{P}_2$ ($\Rightarrow$ LT)

Why? 1) Let $S$ be a (finite) spanning subset of $V$.

Let $S$ be some subset $p \subseteq S$ must be a basis of $V$.

So $|S| > |p| = N$ (since $\dim(V) = N$).

2) If $|S| = N$, then $|S| = |p|$ ($\Rightarrow$ LT), so $S = p$ (since $S \subseteq p$), so $S$ is a basis.

(So the dimension is a useful # which tells us a lot about our V's.)

III - Dimension of Subspaces

Let's now see what we can say about subspaces.

Claim 1) If $V$ is finite-dimensional and $W$ is a subspace of $V$, then $W$ is finite-dimensional and $\dim(W) \leq \dim(V)$.

Claim 2) If $\dim(V) = \dim(W)$, then $W = V$ (3-Dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$).

Why? 1) See below (start adding LT vectors in $W$ cannot add more than $\dim(V)$ vectors)

2) Let $p$ be a basis for $W$.

Then $p$ is a LT subset of $W \subseteq V$ with $\dim(W) = \dim(V)$ vectors.
So \( B \) is a basis of \( V \), so \( W = \text{span}(B) = V \).

**IV - The Basis Extension Theorem**

Lastly, here's one last application of the Replacement Theorem, which leads (in my opinion) to the most important fact in Linear Algebra.

**Corollary** If \( V \) is finite-dimensional (\( \dim(V) = N \)) , then any \( L.I. \) subset of \( V \) can be extended to a basis of \( V \).

**Example** \( \{ (1,1,0), (0,1,1) \} \) is \( L.I. \), so can extend to a basis of \( \mathbb{R}^3 \) \( \{ (1,1,0), (0,1,1), (1,0,1) \} \) of \( \mathbb{R}^3 \).

**Picture**

Why? Let \( L \in = \text{any} \ L.I. \subset \text{set of} \ V \ \ (M \text{ vectors}) \) \n\( \text{gen} = \text{any} \ \text{basis of} \ \ V = B = \{ 1 \} \ \ (N \text{ vectors}) \)

1) By Replacement, there is a subset \( H \) of \( \text{gen} = B \) with \( N - M \) vectors such that \( \text{lin} \ (U \cup H) \ \text{spans} \ \ V \)

   \[ M \quad N - M \]

2) \( \text{lin} \ (U \cup H) \) has at most \( M + (N - M) = N \) vectors (at most due to double-counting).

   But since \( \text{lin} \ (U \cup H) \ \text{spans} \ \ V \) and \( \dim(V) = N \), \( \text{lin} \ (U \cup H) \) must have at least \( N \) vectors.

3) \( \text{so} \ \text{lin} \ (U \cup H) \ \text{is a spanning subset of} \ V \ \text{with} \ N \ \text{vectors} \),

   \( \text{so} \ \text{lin} \ (U \cup H) \ \text{is a basis of} \ V \) (that includes \( L \)).
(And without further ado, here is one of the most important theorems in this course. We'll use this over & over again.)

**Ultra-important theorem (basis extension theorem)**

If $W$ is a subspace of a finite-dim VS $V$, then any basis of $W$ can be extended to a basis of $V$.

\[ V \]

\[ W \]

\[ \{ \ldots \} \]

\[ \text{Basis} \]

\[ \text{For } W \]

\[ \text{For } V \]

Why? Let $\{ \ldots \}$ be a basis of $W$.

Then $\{ \ldots \}$ is LI.

So use conclusion to extend $\{ \ldots \}$ to a basis $\{ \ldots \}$ of $V$.

(and with this, let's claim victory on Chapter 1.

Next time, we'll start Chapter 2, which is all about linear transformations.)