FRIDAY, APRIL 12, 2019

LECTURE 6 – BASIS AND DIMENSION (II) (SECTION 1.6)

PREVIOUSLY ON "THAT'S SO PEYAM", WE DISCOVERED THE REPLACEMENT THEOREM, WHICH IS AT THE ♡ OF OUR LA ADVENTURE. TODAY IS ALL ABOUT ITS PROOF AND FURTHER CONSEQUENCES.

I – THE REPLACEMENT THEOREM

THEOREM: LET LIN BE A LI SUBSET OF V WITH M VECTORS.
LET GEN BE A SPANNING SUBSET OF V WITH N VECTORS.
THEN: 1) $M \leq N$
2) THERE IS A SUBSET H OF GEN WITH $N-M$ VECTORS SUCH THAT $LIN \cup H$ SPANS V

PICTURE LIN = □ □ □ (M), GEN = • • • • • (N)
H = • • $(N-M)$
$LIN \cup H$ = □ □ □ • • (N OR LESS)

(INTUITIVELY: ANY LI SET CAN BE EXTENDED TO A SPANNING SET)

PROOF INDUCTION ON $M = |LIN|$

BASE CASE $M=0$ THEN $LIN = \emptyset$ AND CHECK $H = GEN$ WORKS ✓

INDUCTIVE STEP SUPPOSE P_M TRUE, SHOW P_{M+1} TRUE

LET $LIN = \{V_1, \dots, V_{M+1}\}$ AND GEN BE GIVEN, FIND H.

(**IDEA** SHOW □ □ $\overset{V_{M+1}}{□}$ • • SPANS V

KNOW □ □ • • • SPANS V

BUT □ ∈ SPAN{ □ □ • • • } = V, SO • ∈ SPAN{ $\underbrace{□ □ □ • •}^{H}$ }

(WE'RE REPLACING • WITH □)

STEP 1 CONSIDER $L' = \{V_1, \ldots, \underline{V_M}\}$ (LI), GEN AS ABOVE

THEN BY IND HYP.: 1) $M \leq N$

2) THERE IS A SUBSET $H' = \{U_1, \ldots, U_{N-M}\} \subseteq$ GEN

SUCH THAT $L' \cup H' = \{V_1, \ldots, V_M\} \cup \{U_1, \ldots, U_{N-M}\}$ SPANS V (*)

STEP 2 CONSIDER $V_{M+1} \in V$

BY (*), THERE ARE $a_1, \ldots, a_M, b_1, \ldots, b_{N-M} \in F$ SUCH THAT

$$V_{M+1} = a_1 V_1 + \cdots + a_M V_M + b_1 U_1 + \cdots + b_{N-M} U_{N-M} \quad (**)$$

NOTE IF $N = M$, THEN $H' = \emptyset$ SO (**) BECOMES

$V_{M+1} = a_1 V_1 + \cdots + a_M V_M$, WHICH CONTRADICTS $L_N = \{V_1, \ldots, V_{M+1}\}$ LI $\Rightarrow\Leftarrow$

SO $N > M \Rightarrow N \geq M+1 \Rightarrow$ 1) OF P_{M+1} ✓

STEP 3 SIMILARLY, IN (**), $b_1, \ldots, b_{N-M}$ CANNOT BE ALL 0, SO ONE OF THEM, SAY $b_1 \neq 0$.

SO $b_1 U_1 = -a_1 V_1 \cdots - a_M V_M + V_{M+1} - b_2 U_2 \cdots - b_{N-M} U_{N-M}$

$$U_1 = -\frac{a_1}{b_1} V_1 \cdots - \frac{a_M}{b_1} V_M + \frac{1}{b_1} V_{M+1} - \frac{b_2}{b_1} U_2 \cdots - \frac{b_{N-M}}{b_1} U_{N-M}$$

SO $U_1 \in \text{SPAN}\{V_1, \ldots, V_M, V_{M+1}, U_2, \ldots, U_{N-M}\}$ (***) $\in$ SPAN$\{\ldots\}$

LET $H = \{U_2, \ldots, U_{N-M}\} \subseteq \{U_1, \ldots, U_{N-M}\} = H' \subseteq$ GEN

THEN 1) H HAS $N - M - 1 = N - \underline{(M+1)}$ ELEMENTS

2) $L_{IN} \cup H = \{v_1, \dots, v_{m+1}\} \cup \{u_2, \dots, u_{n-m}\}$ spans V

Because $\{v_1, \dots, v_m\} \cup \{u_1, \dots, u_{n-m}\}$ spans V (by (*))
and $u_1 \in \text{span}\{v_1, \dots, v_{m+1}, u_2, \dots, u_{n-m}\}$ (check)

Hence P_{m+1} is true, so P_m is true for all m ■

II - [redacted] Importance of dimension

The reason the replacement theorem is important is not b/c of its statement, but b/c of its consequences. We've already seen one last time (any 2 bases of V have the same # of elements). Here's another one

Corollary Suppose $\dim(V) = n$, then any LI subset of V with n vectors is a basis for V

Ex $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ is a LI subset of $V = M_{2 \times 2}$

with 4 vectors, and $\dim(V) = (2)(2) = 4$, so it's a basis
($\Rightarrow$ spans V)

Why? Let L be a LI subset of V with n vectors

Use replacement with: $L_{IN} = L$ (n vectors)
$GEN = \beta$ (any basis of V, n vectors)

Then there is a subset H of GEN with $n - n = 0$ vectors ($\Rightarrow H = \varnothing$) such that $L_{IN} \cup H$ spans V

So $L_{IN} \cup H = L \cup \varnothing = L$ spans V

So L is LI and spans V, so L is a basis of V ■

(WHAT ABOUT A SPANNING SET W/ N VECTORS)

CLAIM 1) ANY (FINITE) SET THAT SPANS V MUST HAVE $\geq N$ VECTORS
2) ANY SPANNING SET OF V WITH N VECTORS IS A BASIS FOR V

EX $\{x^2, x^2+x, x^2+x+1\}$ SPANS P_2 AND $\dim(P_2) = 3$, SO IT'S A BASIS FOR P_2 ($\Rightarrow$ LI)

WHY? 1) LET S BE A (FINITE) SPANNING SUBSET OF V

LAST TIME: SOME SUBSET $\beta \subseteq S$ MUST BE A BASIS OF V

SO $|S| \geq |\beta| = N$ (SINCE $\dim(V) = N$) ✓

2) IF $|S| = N$, THEN $|S| = |\beta|$ $(= N)$, SO $S = \beta$ (SINCE $S \subseteq \beta$), SO S IS A BASIS ✓ ■

(SO THE DIM IS A USEFUL # WHICH TELLS US A LOT ABOUT OUR VS)

III – DIMENSION OF SUBSPACES

LET'S NOW SEE WHAT WE CAN SAY ABOUT SUBSPACES

CLAIM 1) IF V IS FINITE-DIMENSIONAL AND W IS A SUBSPACE OF V, THEN W IS FINITE-DIM AND $\dim(W) \leq \dim(V)$
2) IF $\dim(V) = \dim(W)$, THEN $W = V$ (3-DIM SUBSPACE OF $\mathbb{R}^3$ IS $\mathbb{R}^3$)

WHY? 1) SEE BOOK (START ADDING LI VECTORS IN W, CANNOT ADD MORE THAN $\dim(V)$ VECTORS)

2) LET β BE A BASIS FOR W
THEN β IS A LI SUBSET OF $W \subseteq V$ WITH $\dim(W) = \dim(V)$ VECTORS

$$\text{SO } \beta \text{ IS A BASIS OF } V, \text{ SO } \underbrace{W}_{\text{BASIS OF } W} = \underbrace{\text{SPAN}(\beta)}_{} = \underbrace{V}_{\text{BASIS OF } V} \quad \blacksquare$$

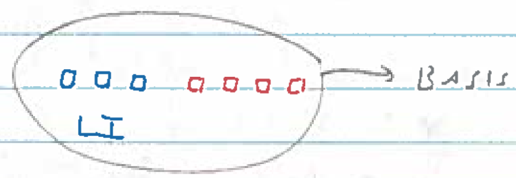
IV — THE BASIS EXTENSION THEOREM

LASTLY, HERE'S ONE LAST APPLICATION OF THE REPLACEMENT THEOREM, WHICH LEADS (IN MY OPINION) TO THE MOST IMPORTANT FACT IN LINEAR ALGEBRA

COROLLARY IF V IS FINITE-DIMENSIONAL ($\dim(V) = N$), THEN ANY LI SUBSET OF V CAN BE EXTENDED TO A BASIS OF V

EX $\{(1,1,0),(0,1,1)\}$ LI, SO CAN EXTEND TO A BASIS, SAY $\{(1,1,0),(0,1,1),\underline{(1,0,1)}\}$ OF $\mathbb{R}^3$

PICTURE



WHY? LET LIN = ANY LI SUBSET OF V (M VECTORS)

GEN = ANY BASIS OF $V = \beta$ (N VECTORS)

1) BY REPLACEMENT, THERE IS A SUBSET H OF GEN $= \beta$ WITH $N-M$ VECTORS SUCH THAT $\underbrace{\text{LIN}}_{M} \cup \underbrace{H}_{N-M}$ SPANS V

2) NOW LIN $\cup$ H HAS <u>AT MOST</u> $M + (N-M) = N$ VECTORS

(AT MOST B/C OF DOUBLE-COUNTING)

BUT SINCE LIN $\cup$ H SPANS V AND $\dim(V) = N$, LIN $\cup$ H MUST HAVE AT <u>LEAST</u> N VECTORS

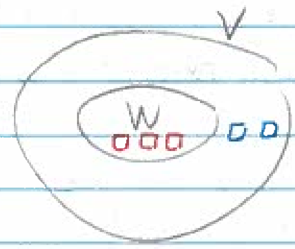
3) SO LIN $\cup$ H IS A SPANNING SUBSET OF V WITH N VECTORS, SO LIN $\cup$ H IS A BASIS OF V (THAT INCLUDES LIN) $\blacksquare$

(AND WITHOUT FURTHER ADO, HERE IS ONE OF THE MOST IMPORTANT THEOREMS IN THIS COURSE, WE'LL USE THIS OVER & OVER AGAIN)

ULTRA-IMPORTANT THEOREM [BASIS EXTENSION THEOREM]

IF W IS A SUBSPACE OF A FINITE-DIM VS V, THEN ANY BASIS OF W CAN BE EXTENDED TO A BASIS OF V

PICTURE



$\beta = \{ \underbrace{\square\,\square\,\square}_{\text{BASIS FOR } W} \;\square\,\square \}$ BASIS FOR V

WHY? LET β' BE A BASIS OF W
THEN β' IS LI
SO USE COROLLARY TO EXTEND β' TO A BASIS β OF V ■

(AND W/ THIS, LET'S CLAIM VICTORY ON CHAPTER 1
NEXT TIME, WE'LL START CHAPTER 2, WHICH IS ALL ABOUT LINEAR TRANSFORMATIONS)