3.1 (a) True.

(b) False. Any vector space needs to contain the 0 vector.

(c) True. Let W be the 0 vector space.

(d) False. The subsets may not contain the zero vector.

(e) True.

(f) False. The trace is the *sum* of the diagonal entries.

(g) False. The zero vector in W is $(0,0,0)$ and the zero vector in $\mathbb{R}^2$ is $(0,0)$.

3.2 (b)

$$\begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix}^t = \begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix}$$

Trace $= -4 + 1 + 5 = 2$.

3.8 (a) Is a subspace.

$$(a_1, a_2, a_3)+(b_1, b_2, b_3) = (a_1+b_1, a_2+b_2, a_3+b_3) \quad a_1+b_1 = 3a_2+3b_2 = 3(a_2+b_2), a_3+b_3 = -a_2+-b_2 = -1(a_2+b_2)$$

$$c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3) \quad ca_1 = c(3a_2) = 3(ca_2), ca_3 = c(-a_2) = -ca_2)$$

The $\vec{0}$ satisfies $0 = 3 \cdot 0$, $0 = -1 \cdot 0$.

(b) Not a subspace, does not contain the zero vector.

(c) Is a subspace. Prove as in (a)

(d) Is a subspace. Prove as in (a)

(e) Not a subspace, does not contain the zero vector.

(f) Not a subspace. $(0, 2, \sqrt{2})$, and $(0, -2, \sqrt{2})$ are in W_6, but their sum $(0, 0, 2\sqrt{2})$ is not in W_6.

3.9 (a) $W_1 \cap W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2, a_3 = -a_2, 2a_1 - 7a_2 + a_3 = 0\} = \{$ the zero vector $\}$

(b) $W_1 \cap W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2, a_3 = -a_2, a_1 - 4a_2 - a_3 = 0\} = W_1$

(c) $W_3 \cap W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3, 2a_1 - 7a_2 + a_3 = 0\} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 3a_1 - 11a_2 = 0, a_3 = a_1 - 4a_2\}$

3.13 Let $W = \{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$. The zero vector sends every element to 0 so the zero vector is in W. Given $f, g \in W$, $(f + g(s_0)) = f(s_0) + g(s_0) = 0 + 0 = 0$, so $f + g \in W$ and W is closed under addition. Given $c \in F$, $(cf)(s_0) = c(f(s_0)) = c \cdot 0 = 0$, so $cf \in W$ and W is closed under scalar multiplication. Therefore W is a subspace for any choice of s_0.

3.19 Let W_1, W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of $V \iff W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

$\Rightarrow$ Given a vector $w_1 \in W_1$, and $w_2 \in W_2$, consider $v := w_1 + w_2$. Since $W_1 \cup W_2$ is a subspace, $v \in W_1 \cup W_2$.

Case 1: There is some $w_2 \in W_2$ such that $v := w_1 + w_2$ is in W_2. Since $w_1 = v + (-w_2)$ and $v, -w_2\ in W_2$, w_1 is also in W_2. Therefore $W_1 \subseteq W_2$.

Case 2: For every $w_2 \in W_2$, $v := w_1 + w_2$ is in W_1. Since $w_2 = v + (-w_1)$, and $w_1, v \in W_1$, w_2 is also in W_1 for all $w_2 \in W_2$. Therefore $W_2 \subseteq W_1$.

$\Leftarrow$ If $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, $W_1 \cup W_2$ is equal to W_2 or W_1 respectively. W_1 and W_2 are subspaces so the union is also a subspace.

3.23 $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$

(a) The zero vector is in W_1 and W_2 so $\vec{0} + \vec{0} = \vec{0}$ is in $W_1 + W_2$. For any $v, v' \in W_1 + W_2$ and $c \in F$, there exists $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$ such that $v = w_1 + w_2$ and $v' = w_1' + w_2'$.

$$v + v' = w_1 + w_2 + w_1' + w_2' = (w_1 + w_1') + (w_2 + w_2'), \quad w_1 + w_1' \in W_1, w_2 + w_2' \in W_2,$$

therefore $v + v' \in W_1 + W_2$.

$$c \cdot v = c(w_1 + w_2) = cw_1 + cw_2, \quad cw_1 \in W_1, cw_2 \in W_2,$$

therefore $c \cdot v$ is in $W_1 + W_2$. Thus, $W_1 + W_2$ is a subspace. $W_1 \subset W_1 + W_2$ because $w_1 + 0 \in W_1 + W_2$ for any $w_1 \in W_1$. Similarly, $W_2 \subset W_1 + W_2$.

(b) Let W be a vector space which contains W_1 and W_2. Then for all vectors $w_1 \in W_1$ and $w_2 \in W_2$, $w_1 + w_2$ must be in W. Therefore $W_1 + W_2 \subset W$.

4.1 (a) True.

(b) False. The span of the empty set is the zero vector.

(c) True.

(d) False.

(e) True.

(f) False.

1.4.2 (a)

$$\begin{array}{rcrcrcrcr} 2x_1 & - & 2x_2 & - & 3x_3 & & & = & -2 \\ 3x_1 & - & 3x_2 & - & 2x_3 & + & 5x_4 & = & 7 \\ x_1 & - & x_2 & - & 2x_3 & - & x_4 & = & -3 \end{array} \longmapsto \begin{array}{rcrcrcrcr} x_1 & - & x_2 & & & + & 3x_4 & = & 5 \\ & & & & x_3 & + & 2x_4 & = & 4 \\ & & & & & & 0 & = & 0 \end{array}$$

$(5 + x_2 + -3x_4, x_2, 4 - 2x_4, x_4)$ are solutions to the linear set of equations for any $x_2, x_4 \in \mathbb{R}$

(f)

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 6x_3 & = & -1 \\ 2x_1 & + & x_2 & + & x_3 & = & 8 \\ 3x_1 & + & x_2 & - & x_3 & = & 15 \\ x_1 & + & 3x_2 & + & 10x_3 & = & -5 \end{array} \longmapsto \begin{array}{rrrcr} x_1 & & & = & 3 \\ & x_2 & & = & 4 \\ & & x_3 & = & -2 \\ & & 0 & = & 0 \end{array}$$

$(3, 4, -2)$ is the only solution to this set of linear equations.

1.4.3 (a) $(2, 4, -1) = \frac{-1}{3} \cdot (-2, 0, 3) + \frac{4}{3} \cdot (1, 3, 0)$.

(f) $(-3, -3, 3) = \frac{1}{2} \cdot (-2, 2, 2) + -2 \cdot (1, 2, -1)$.

1.4.4 (d) $1/5(x^3 + x^2 + 2x + 13) + 2/5(2x^3 - 3x^2 + 4x + 1) = x^3 - x^2 + 2x + 3$

1.4.7 $(a_1, a_2, \ldots, a_n) = a_1 \cdot e_1 + a_2 \cdot e_2 + \cdots + a_n \cdot e_n$, so every vector in F^n can be written as a linear combination of the e_i's.

1.4.14 **Step 1:** $\operatorname{span}(S_1 \cup S_2) \subset \operatorname{span}(S_1) + \operatorname{span}(S_2)$

Take $u \in \operatorname{span}(S_1 \cup S_2)$, i.e. $u = \sum c_i \cdot u_i$, where $u_i \in S_1 \cup S_2$. Split the sum into two pieces, one where $u_i \in S_1$ and the other where $u_1 \notin S_1$, i.e. $u = \sum_{u_i \in S_1} c_i u_i + \sum_{u_j \notin S_1} c_j u_j$. Then the first summation is in the span of S_1, and since all u_i were in $S_1 \cup S_2$, the second summation is in the span of S_2. Thus we have the desired containment.

Step 2: $\operatorname{span}(S_1) + \operatorname{span}(S_2) \subset \operatorname{span}(S_1 \cup S_2)$

Take $u \in \operatorname{span}(S_1) + \operatorname{span}(S_2)$, so $u = v + w$ where $v \in \operatorname{span}(S_1)$, $w \in \operatorname{span}(S_2)$. Then $v + w$ is a linear combination of vectors in S_1 or S_2, so $u = v + w$ is in the span of $S_1 \cup S_2$.

Section 1.5: 1,3,5,7,10,11,15; Section 1.6: 1,3,5,12,15

1.5.1 (a) False. There exists a vector in S which is a linear combination of other vectors in S.

(b) True. $c \cdot 0_V = 0_V$ for any nonzero $c \in F$.

(c) False. The empty set is linearly independent.

(d) False. The set consisting of one nonzero vector is linearly independent.

(e) True.

(f) True.

1.5.3

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{M_{3\times 2}(F)}$$

1.5.5 Assume $a_0 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0_{P_n(F)}$. Since the zero vector has negative degree, $a_i = 0$ for all i. Therefore $\{1, x, x^2, \ldots, x^n\}$ is linearly independent.

1.5.7

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

generate the 2×2 diagonal matrices.

1.5.10 $\{(1,0,0), (0,1,0), (1,1,0)\}$ are a linearly dependent set and none are a multiple of the other.

1.5.11 Span(S) $= \{a_1 u_+ \cdots + a_n u_n | a_i \in Z_2\}$ and because S is a linearly independent set, each vector in Span(S) can be written uniquely in this way. Therefore to count the number of vectors in Span(S), we need to count the possibilities for $(a_1, \ldots, a_n)$. Since Z_2 has 2 elements, there are 2^n possibilities for $(a_1, \ldots, a_n)$, and therefore 2^n vectors in Span(S).

1.5.15 $S = \{u_1, u_2, \ldots, u_n\}$ Prove S is linearly dependent $\iff$ $u_1 = 0$ or $u_{k+1} \in$Span$(\{u_1, \ldots, u_k\})$

$\Leftarrow$ This direction follows from the definitions

$\Rightarrow$ Since S is linearly dependent, there exists $a_1, \ldots, a_n \in F$ not all zero, such that $a_1 u_1 + \cdots + a_n u_n = 0$. Let k be such that $a_i = 0$ for all $i > k+1$ and $a_{k+1} \neq 0$. Then our equation reduces to $a_1 u_1 + \cdots + a_{k+1} u_{k+1} = 0$. If $k = 0$, then $u_1 = 0$. If $k > 0$, then $a_1 u_1 + \cdots + a_k u_k = a_{k+1} u_{k+1}$. Since $a_{k+1} \neq 0$, we can divide by a_{k+1} and $u_{k+1} \in$Span$(\{u_1, \ldots, u_k\})$.