MONDAY, APRIL 15, 2019

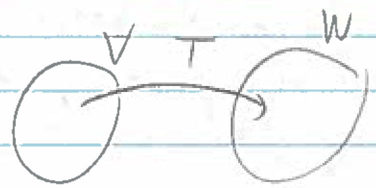
LECTURE 7 – LINEAR TRANSFORMATIONS (I)

HELLO AND WELCOME TO CHAPTER 2! WHILE IN CHAPTER 1 WE STUDIED VS AS OBJECTS ON THEIR OWN, IN THIS CHAPTER WE WANT TO STUDY THE RELATIONSHIP BETWEEN VS, AND THE BEST WAY TO DESCRIBE THAT IS WITH LINEAR TRANSFORMATIONS.

I – DEFINITION AND EXAMPLES

DEF LET V AND W BE VS AND $T: V \to W$ BE A FUNCTION FROM V TO W. THEN T IS A LINEAR TRANSFORMATION (LT) IF

1) FOR ALL x AND y IN V, $T(x+y) = T(x) + T(y)$
2) FOR ALL x IN V AND c IN $\mathbb{F}$, $T(cx) = cT(x)$



(SO A LT IS A SPECIAL KIND OF FUNCTION, ONE THAT RESPECTS ADDITION AND SCALAR MULT)

⚠ HERE W IS NOT A SUBSPACE OF V!

EX 1 $T: \underset{V}{\mathbb{R}^3} \to \underset{W}{\mathbb{R}^2}$ $\quad T(a_1, a_2, a_3) = (a_1 - 3a_2, 2a_1 + 4a_3)$

EX 2 $T: C([0,1]) \to \mathbb{R}$

↳ CONT. FUNCTIONS FROM [0,1] TO $\mathbb{R}$

$$T(f) = \int_0^1 f(x)\,dx$$

WHY? 1) $T(f+g) = \int_0^1 (f+g)(x)\,dx = \int_0^1 f(x) + g(x)\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx$

$= T(f) + T(g)$ ✓

2) $T(cf) = \int_0^1 (cf)(x)\,dx = c\int_0^1 f(x)\,dx = cT(f)$ ✓

Non-Ex 3 $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(a_1,a_2) = (a_1^2, a_2)$ B/C

$$T(2,0) = (2^2,0) = (4,0) \text{ BUT } 2T(1,0) = 2(1,0) = (2,0)$$

Non-Ex 4 $T: \mathbb{R}^2 \to \mathbb{R}^3$ $T(a_1,a_2) = (a_1,a_2,1)$

FACT $T(0_V) = 0_W$ $\quad 0_V = $ 0-VECTOR IN V
$\quad 0_W = $ 0-VECTOR IN W

WHY? $T(0_V) = T(0\,0_V) = 0\,T(0_V) = 0_W$

HERE $T(\underbrace{0,0}_{0_V}) = (0,0,1) \neq (\underbrace{0,0,0}_{0_W})$

(IT TURNS OUT THERE'S A QUICKER WAY TO CHECK IF T IS LINEAR)

CLAIM T IS LINEAR $\iff$ $T(cx+y) = cT(x) + T(y)$ $(*)$
FOR ALL $x, y \in V$, $c \in \mathbb{F}$

$(\Rightarrow)$ $T(cx+y) = T(cx) + T(y) = cT(x) + T(y)$

$(\Leftarrow)$ $T(x+y) = T(1x+y) \underset{(*) \text{ WITH } c=1}{=} 1T(x) + T(y) = T(x) + T(y)$ ✓

$$T(cx) = T(cx + 0_V) \underset{(*) \text{ WITH } y = 0_V}{=} cT(x) + \underbrace{T(0_V)}_{\text{CHECK } = 0_W} = cT(x) + 0_W = cT(x) \checkmark$$

EX 5 $T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ $T(y) = y'' - 4y$

$$\begin{aligned}
T(cy_1 + y_2) &= (cy_1 + y_2)'' - 4(cy_1 + y_2) \\
&= cy_1'' + y_2'' - 4cy_1 - 4y_2 \\
&= c(y_1'' - 4y_1) + (y_2'' - 4y_2) \\
&= cT(y_1) + T(y_2) \checkmark
\end{aligned}$$

EX 6 $I_V: V \to V$, $I_V(x) = x$ IDENTITY TRANSF.

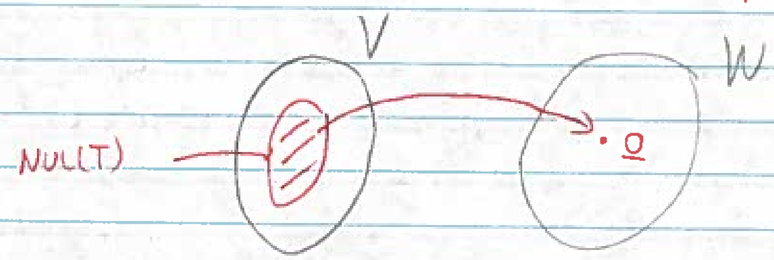
EX 7 $T_0: V \to W$ $T_0(x) = \underline{0}$ ZERO TRANSF.

II – NULLSPACE

(IN FACT, RELATED TO THIS IS THE CONCEPT OF A NULLSPACE)

DEF IF $T : V \to W$ IS LINEAR, THEN

$$N(T) = NUL(T) = \{ x \text{ IN } \underline{\underline{V}} \mid T(x) = \underline{0} \}$$ (NULLSPACE / KERNEL OF T)



EX $T : P_2(\mathbb{R}) \to \mathbb{R}^2 \qquad T(p) = (p(0), p'(0))$

SUPPOSE $p = a_0 + a_1 x + a_2 x^2 \in N(T)$

THEN $T(p) = (p(0), p'(0)) = (0, 0) \leftarrow \underline{0}$

BUT $p(0) = 0 \Rightarrow a_0 + a_1(0) + a_2(0)^2 = 0 \Rightarrow \underline{a_0 = 0}$

AND $p'(x) = a_1 + 2a_2 x$, SO $p'(0) = a_1 + 2a_2(0) = a_1 = 0$,

SO $\underline{a_1 = 0}$

SO $p(x) = 0 + 0x + a_2 x^2 = a_2 x^2$

$N(T) = \{ a_2 x^2 \mid a_2 \in \mathbb{R} \} = \text{SPAN}\{x^2\}$

(NOTICE $N(T)$ IS A VS HERE; NOT A COINCIDENCE, THIS IS ALWAYS TRUE)

FACT $N(T)$ IS A SUBSPACE OF $\underline{\underline{V}}$

WHY? EX CHECK +: SUPPOSE X & Y ARE IN $N(T)$, SO $T(x) = \underline{0}$ AND $T(y) = \underline{0}$

BUT THEN $T(x+y) = T(x) + T(y) = \underline{0} + \underline{0} = \underline{0}$,

SO $T(x+y) = \underline{0}$, SO $x+y \in N(T)$ ✓

EX SHOWED $T(y) = y'' - 4y$ IS LINEAR,

SO SOLUTIONS TO $\underbrace{y'' - 4y = 0}_{N(T)}$ IS A SUBSPACE OF $C^\infty(\mathbb{R})$
(W/O EVEN SOLVING IT!)

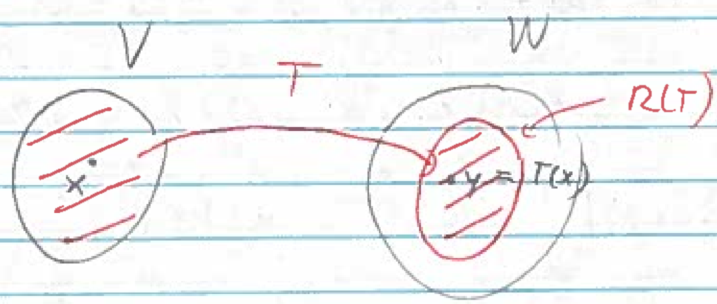
III — RANGE

(ON THE OTHER SIDE OF THE SPECTRUM IS THE NOTION OF RANGE)

DEF IF $T : V \to W$ IS LINEAR, THEN

$$R(T) = \{ T(x) \mid x \in V \}$$ (= SET OF ALL POSSIBLE OUTPUTS OF V)

EQUIVALENTLY $y \in R(T) \iff y = T(x)$ FOR SOME $x \in V$



(IF T IS AN AIRPLANE, THINK OF $R(T)$ AS THE SET OF ALL POSSIBLE COUNTRIES YOU CAN FLY TO)

EX $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(a_1, a_2) = (a_1, a_2, 0)$

$\text{RAN}(T) = xy\text{-PLANE IN } \mathbb{R}^3$

$(1,2,0) \in R(T)$ B/C $(1,2,0) = T(1,2)$
$\qquad\qquad\qquad\qquad y \qquad\qquad x$

BUT $(1,2,1) \notin R(T)$ B/C $(1,2,1)$ IS NOT OF THE FORM $(a_1, a_2, 0)$

(WHY IMP?)

FACT $R(T)$ IS A SUBSPACE OF W

WHY? EX CHECK + : SUPI

IF y_1 & y_2 ARE IN $R(T)$, THEN $y_1 = T(x_1)$, $y_2 = T(x_2)$
FOR SOME $x_1, x_2 \in V$

THEN $y_1 + y_2 = T(x_1) + T(x_2) = T(x_1 + x_2) = T(\text{BLAH})$,
SO $y_1 + y_2 \in R(T)$

HOW TO FIND $R(T)$?

FACT IF $\beta = \{V_1, \dots, V_N\}$ IS A BASIS OF V, THEN

$R(T) = \text{SPAN}\{T(V_1), \dots, T(V_N)\}$

EX $T: M_{2\times 2} \to M_{2\times 2}$ $\quad T(A) = A + A^T$ (SYMMETRIZATION OF A)

EX $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 4 \end{bmatrix}$

$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ IS A BASIS FOR $M_{2\times 2}$

SO $R(T) = \text{SPAN}\left\{ T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

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$$= \text{SPAN}\left\{ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \cancel{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

$$= \left\{ a_1 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \;\middle|\; a_1, a_2, a_3 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 2a_1 & a_2 \\ a_2 & 2a_3 \end{bmatrix} \;\middle|\; a_1, a_2, a_3 \in \mathbb{R} \right\}$$

= 2X2 SYMMETRIC MATRICES