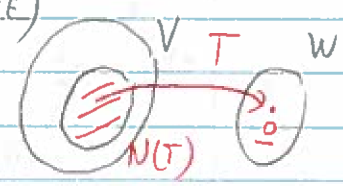
WEDNESDAY, APRIL 17, 2019

LECTURE 8 – LINEAR TRANSFORMATIONS (II) (SECTION 2.1)

☺ HAPPY W AND LET'S CONTINUE OUR EXPLORATION OF LT, STARTING WITH THE SINGLE MOST IMPORTANT THEOREM ABOUT LT, THE:

I – RANK – NULLITY THEOREM / DIMENSION THEOREM

(PREVIOUSLY ON "GAME OF PEYAMS", WE DISCOVERED TWO IMPORTANT SUBSPACES RELATED TO A LT, THE NULLSPACE AND THE RANGE)



DEF $N(T) = \{ x \in V \mid T(x) = 0 \}$

$R(T) = \{ T(x) \mid x \in V \}$

NOTE $\dim(N(T)) = \text{NULLITY}(T)$

$\dim(R(T)) = \text{RANK}(T)$

(ONE OF THE TRUE MIRACLES OF LA IS THAT THESE TWO NOTIONS ARE JUST TWO ≠ SIDES OF THE SAME COIN!)

THEOREM [DIMENSION THEOREM / RANK-NULLITY THEOREM]

(IF $\dim(V) < \infty$ AND $T: V \to W$ LINEAR, THEN)

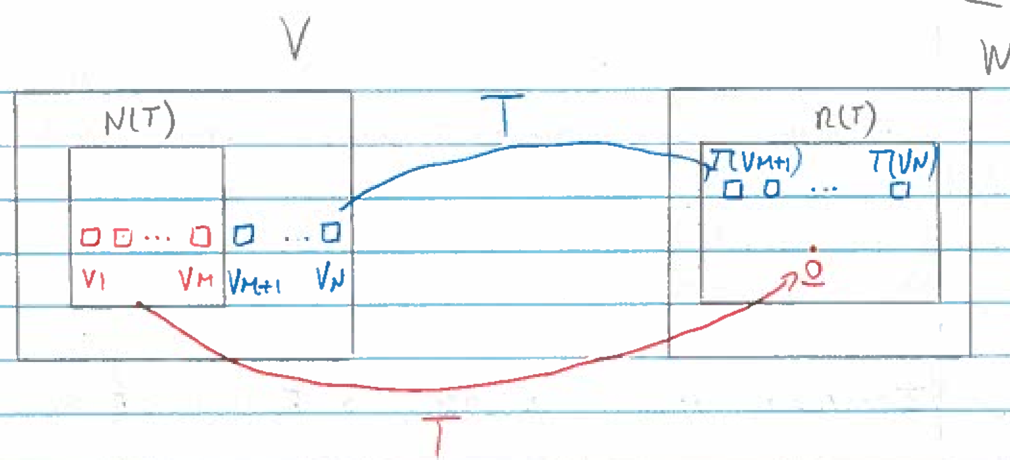
$$\dim(\text{NUL}(T)) + \text{RANK}(T) = \dim(V)$$

(INTUITIVELY: $N(T)$ MEASURES HOW BAD T IS, $R(T)$ MEASURES HOW GOOD T IS } THIS SAYS THEY BALANCE OUT)

PROOF ⚠ ONE OF MY FAVORITE PROOFS!

1) LET $\{v_1, \dots, v_m\}$ BE A BASIS FOR $N(T)$ $(m = \dim(N(T)))$

EXTEND $\{v_1, \dots, v_m\}$ TO A BASIS $\{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$ OF V



(IDEA) $v_1, \dots, v_m$ GET SENT TO 0, EVERYTHING ELSE GETS SENT TO $R(T)$

2) CLAIM $\{T(v_{m+1}), \dots, T(v_n)\}$ IS A BASIS FOR $R(T)$

NOTE THEN WE'RE DONE B/C:

$$\text{RANK}(T) = \dim(R(T)) \overset{\text{BY CLAIM}}{=} n-m = \dim(V) - \dim(N(T)) \checkmark$$

3) PROOF OF CLAIM

SPAN LET $y \in R(T)$, THEN $y = T(x)$ FOR SOME $x \in V$

SINCE $\{v_1, \dots, v_n\}$ IS A BASIS FOR V, $x = a_1 v_1 + \dots + a_n v_n$ FOR $a_1, \dots, a_n \in F$

THEN $y = T(x) = T(a_1 v_1 + \dots + a_n v_n)$

$v_1, \dots, v_m \in N(T)$ $\quad = a_1 \cancel{T(v_1)}^0 + \dots + a_m \cancel{T(v_m)}^0 + a_{m+1} T(v_{m+1}) + \dots + a_n T(v_n)$

$= a_{m+1} T(v_{m+1}) + \dots + a_n T(v_n)$

SO $y \in \text{SPAN}\{T(v_{m+1}), \dots, T(v_n)\}$ ✓

LI SUPPOSE $a_{m+1} T(v_{m+1}) + \dots + a_n T(v_n) = 0$ (SHOW $a_{m+1}, \dots, a_n = 0$)

THEN $T(a_{m+1} v_{m+1} + \dots + a_n v_n) = 0$

SO $a_{m+1} v_{m+1} + \dots + a_n v_n \in N(T) = \text{SPAN}\{v_1, \dots, v_m\}$

SO $a_{m+1} v_{m+1} + \dots + a_n v_n = a_1 v_1 + \dots + a_m v_m$ FOR SOME $a_1, \dots, a_m$

SO $-a_1 v_1 - \dots - a_m v_m + a_{m+1} v_{m+1} + \dots + a_n v_n = 0$

BUT $\{V_1, \dots, V_N\}$ IS A BASIS SO $-a_1 = 0, \dots, -a_m = 0, \underbrace{a_{m+1} = 0, \dots, a_n = 0}_{\checkmark}$

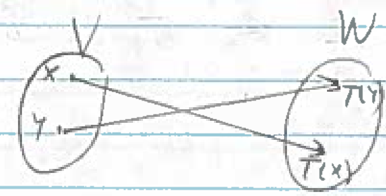
(WILL SEE SOON WHY IT'S USEFUL)

II – ONE-TO-ONE AND ONTO

(HERE'S THE NEAT THING: SINCE LT ARE FUNCTIONS, EVERYTHING WE KNOW ABOUT FUNCTIONS (FROM MATH 13) WE CAN ALSO APPLY TO LT. IN PARTICULAR, IT MAKES SENSE TO TALK ABOUT 1-1 AND ONTO LT)

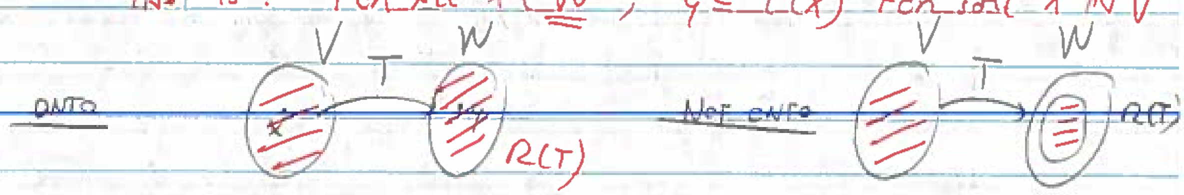
RECALL: 1) T IS 1-1 IFF $T(x) = T(y) \Rightarrow x = y$

(EQUIVALENTLY: $x \neq y \Rightarrow T(x) \neq T(y)$)



2) T IS ONTO W IFF $R(T) = W$

THAT IS: FOR ALL $y \in W$, $y = T(x)$ FOR SOME x IN V



(IT TURNS OUT THAT FOR LT, THERE'S A MUCH EASIER WAY TO SHOW SOMETHING IS 1-1):

FACT: T IS 1-1 $\Longleftrightarrow N(T) = \{0\}$ $\quad (T(x) = 0 \Rightarrow x = 0)$

WHY? $(\Rightarrow)$ IF T IS 1-1 AND $T(x) = 0$, THEN

$T(x) = 0 = T(0)$, SO $x = 0$

$(\Leftarrow)$ IF $T(x) = T(y)$ AND $N(T) = \{0\}$, THEN $T(x) - T(y) = 0$

so $T(x-y)=\underline{0}$, so $x-y=\underline{0}$, so $x=y$ ✓

EX $T: P_2 \to \mathbb{R}^2$, $T(p)=(p(0), p'(0))$

SHOWED $N(T)=\text{SPAN}\{x^2\} \neq \{0\}$, so T IS NOT 1-1

BUT $\text{RANK}(T) = \text{DIM}(P_2) - \text{DIM}(N(T)) = 3 - 1 = 2$,

so $R(T)$ IS A 2 DIM SUBSPACE OF $\mathbb{R}^2$, so $R(T)=\mathbb{R}^2$

so T IS ONTO $\mathbb{R}^2$.

(IN THIS EX, 1-1 $\neq$ ONTO, BUT THE MIRACLE OF LA IS: IF V AND W HAVE THE SAME DIM, THEN 1-1 = ONTO!)

COOL FACT IF $\text{DIM}(V) = \text{DIM}(W) < \infty$ AND $T: V \to W$, THEN

T IS 1-1 $\Leftrightarrow$ T IS ONTO W

WHY? BY RANK-NULLITY, $\text{DIM}(N(T)) + \text{RANK}(T) = \text{DIM}(V)$

($\Leftarrow$) IF T IS ONTO W, $R(T)=W$, so $\text{RANK}(T) = \text{DIM}(R(T)) = \text{DIM}(W)$

so $\text{DIM}(N(T)) = \text{DIM}(V) - \text{RANK}(T) = \text{DIM}(V) - \text{DIM}(W) = 0$

so $N(T)=\{\underline{0}\}$, so T IS 1-1 ✓

($\Rightarrow$) IF T IS 1-1, $N(T)=\{\underline{0}\}$, so $\text{DIM}(N(T))=0$, so

$\text{RANK}(T) = \text{DIM}(V) - \overset{0}{\cancel{\text{DIM}(N(T))}} = \text{DIM}(V)$

so $R(T)$ IS A $\text{DIM}(V)=\text{DIM}(W)$ -DIMENSIONAL SUBSPACE OF W

so $R(T)=W$ AND T IS ONTO W ●

⚠ IMPORTANT THAT V IS FINITE-DIM!

EX $T: \underset{V}{C^\infty(\mathbb{R})} \to \underset{W}{C^\infty(\mathbb{R})}$, $T(f)=f'$

CAN CHECK T IS ONTO BUT NOT 1-1

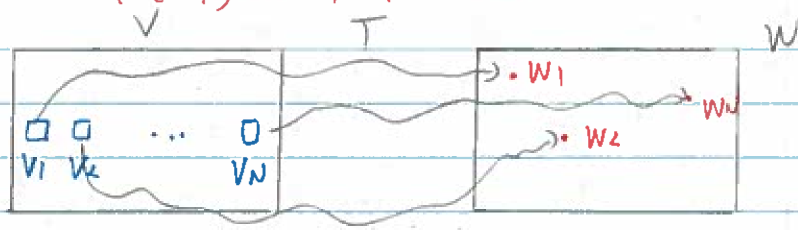
III – EXISTENCE OF LT

(LASTLY, YOU MAY ASK: DO LT EVEN EXIST? AND IS IT EASY TO CONSTRUCT LT? THANKFULLY THE ANSWER IS YES AND YES, AND THIS IS SUPER IMPORTANT FOR NEXT TIME)

FACT IF $\{v_1, \ldots, v_n\}$ IS A BASIS FOR V AND $\{w_1, \ldots, w_n\}$ ARE ANY VECTORS IN W,

THEN THERE IS A UNIQUE LT $T: V \to W$ SUCH THAT $T(v_i) = w_i$ FOR ALL i

PICTURE



EX $\beta = \{\underset{v_1}{(1,1)}, \underset{v_2}{(1,-1)}\}$ IS A BASIS FOR $\mathbb{R}^2$, SO THERE IS A UNIQUE LT $T: \mathbb{R}^2 \to \mathbb{R}^3$ SUCH THAT $T(\underset{v_1}{1,1}) = \underset{w_1}{(2,0,2)}$ AND $T(\underset{v_2}{1,-1}) = (4,0,4)$, NAMELY $T(x,y,z) = (3x-y, 0, 3x-y)$

REMARKS 1) THIS SAYS: TO FIND T, ENOUGH TO SPECIFY T ON BASIS VECTORS.

2) UNIQUE MEANS:

FACT IF S AND $T: V \to W$ SATISFY $S(v_i) = T(v_i)$ FOR ALL i, THEN $S = T$

3) IDEA OF PROOF: IF $x \in V$, $x = a_1 v_1 + \cdots + a_n v_n$

THEN $T(x)$ SHOULD BE $a_1 T(v_1) + \cdots + a_n T(v_n)$

$= a_1 w_1 + \cdots + a_n w_n$

SO DEFINE T BY: $T(x) = a_1 w_1 + \cdots + a_n w_n$ WHENEVER $x = a_1 v_1 + \cdots + a_n w_n$

(<u>SHOW</u> 1) MAKES SENSE, INDEPENDENT OF CHOICE OF a_i

2) T LINEAR

3) $T(v_i) = w_i$

4) T IS UNIQUE)