AP SOLUTIONS

(HW #3)

AP1 LET $N = |S_1| = |S_2|$

USING THE REPLACEMENT THEOREM WITH

$LIN = S_1$ (LI)

$GEN = S_2$ (SPANS V)

WE GET THAT THERE IS A SUBSET H OF S_2 WITH $|S_2| - |S_1| = 0$ ELEMENTS SUCH THAT

$LIN \cup H$ SPANS V

BUT SINCE H HAS NO ELEMENTS, $H = \emptyset$, SO

$LIN \cup H = LIN \cup \emptyset = LIN = S_1$ SPANS V

AP2 $(\Rightarrow)$ SUPPOSE T IS LINEAR, SHOW G IS A SUBSPACE OF $V \times W$

1) <u>ZERO VECTOR</u> LETTING $V = 0_V$, WE GET

$$(0_V, T(0_V)) \in G, \text{ SO } \underbrace{(0_V, 0_W)}_{\text{ZERO VECTOR OF } V \times W} \in G$$

2) <u>ADDITION</u> SUPPOSE $x = (v, T(v))$ AND $y = (w, T(w)) \in G$ $(v, w \in V)$

THEN $x + y = (v, T(v)) + (w, T(w))$

$$= (v + w, T(v) + T(w))$$

$$= (v+w, T(v+w))$$

$$= (z, T(z)) \in G \quad (\text{WITH } z = v+w \in V)$$

3) <u>SCALAR MULT</u> IF $x = (v, T(v)) \in G$ AND $c \in \mathbb{F}$, THEN

$$cx = c(v, T(v)) = (cv, cT(v)) = (cv, T(cv))$$

$$= (z, T(z)) \in G \quad (\text{WITH } z = cv \in V)$$

HENCE G IS A SUBSPACE OF $V \times W$ ✓

$(\Leftarrow)$ SUPPOSE G IS A SUBSPACE OF $V \times W$, SHOW T IS LINEAR

1) LET $u, v \in V$, THEN $(u, T(u))$ AND $(v, T(v))$ ARE IN G, SO

$(u, T(u)) + (v, T(v))$ IS IN G, SO

$(u, T(u)) + (v, T(v)) = (u+v, T(u) + T(v)) = (z, T(z))$ FOR SOME $z \in V$

$z = u+v$

BUT THEN $z = u + v$ AND $T(z) = T(u) + T(v) \Rightarrow T(u+v) = T(u) + T(v)$ ✓

2) LET $u \in V$, $c \in \mathbb{F}$, THEN $(u, T(u))$ IS IN G, SO

$c(u, T(u)) \in G$, SO

$c(u, T(u)) = (cu, cT(u)) = (z, T(z))$ FOR SOME $z \in V$

SO $z = cu$ AND $T(cu) = T(z) = cT(u)$ ✓

SO T IS LINEAR ∎