6.1 (a) False. The empty set is a basis for the zero vector space.

(b) True.

(c) False. Infinite dimensional vector spaces do not have finite bases.

(d) False. $\{(1,0),(0,1)\}$ and $\{(1,0),(1,1)\}$ are both bases for $\mathbb{R}^2$.

(e) True.

(f) False. The dimension is $n+1$.

(g) False. The dimension is mn.

(h) True.

(i) False.

(j) True.

(k) True.

(l) True.

6.3 (b), (c), and (d) are bases. (a) and (e) are not.

6.5 No. A set of 4 vectors in a 3-dimensional space is never linearly independent.

6.8 u_1, u_3, u_5, and u_7 form a basis for W.

6.12 Since V is 3-dimensional, it suffices to show that $\{u+v+w, v+w, w\}$ is linearly independent.

$$a(u+v+w)+b(v+w)+cw=0=au+(a+b)v+(a+b+c)w.$$

Since $\{u,v,w\}$ are linearly independent, $a=a+b=a+b+c=0$. Therefore $a=b=c=0$, and $\{u+v+w, v+w, w\}$ are linearly independent.

6.15 The set $\{E_{i,j}, E_{1,1}-E_{k,k} : 1<k\leq n, 1\leq i,j\leq n, i\neq j\}$ is a basis for the trace zero matrices. ($E_{i,j}$ is the matrix with 1 in the i,j place and 0's everywhere else.) The dimension is n^2-1 because there are n^2-1 basis vectors.

1· 6.23 (a) $v \in \text{Span}(\{v_1, \cdots, v_k\}) \iff \dim(W_1) = \dim(W_2)$

(b) If $\dim(W_1) \neq \dim(W_2)$, then $\dim(W_1) + 1 = \dim(W_2)$

Proof: Since we have only added one more vector to a generating set of W_2, the dimension can at most increase by one, but we've assumed they aren't equal, so it must increase by at least one. Therefore $\dim(W_2) = \dim(W_1) + 1$.

1· 6.26 The dimension of the subspace $W = \{f \in P_n(R) : f(a) = 0\}$ is n. We prove this by computing a basis. A basis for this space is $\{x^i - a^i : 1 \leq i \leq n\}$. One can check that this set is linearly independent, has n elements, and the span is contained inside W. If W was bigger, then W would have dimension $n+1$ and be equal to $P_n(R)$, but W is strictly contained in $P_n(R)$. Therefore this set is a basis for W.

1· 6.28 If $\{v_1, \ldots, v_n\}$ is a basis for V over $\mathbb{C}$, then $\{v_1, iv_1, v_2, iv_2, \ldots, v_n, iv_n\}$ is a basis for V over (R).

Proof: Linearly Independent. Say $a_1v_1 + b_1iv_1 + \cdots + a_nv_n + a_niv_n = 0$. Then by regrouping we get $(a_1 + ib_1)v_1 + \cdots + (a_n + ib_n)v_n = 0$, but $v_1, \ldots, v_n$ are linearly independent over $\mathbb{C}$ so $a_i + ib_i = 0$ which implies that $a_i = b_i = 0$.

Generate. Take any $v \in V$. Over $\mathbb{C}$, we can write this as $v = (a_1 + ib_1)v_1 + \cdots + (a_n + ib_n)v_n$ for some $a_i + ib_i \in \mathbb{C}$. By distributing we see $v = a_1v_1 + b_1iv_1 + \cdots + a_nv_n + a_niv_n$, so the set generates.

1· 6.29 (a) Start with a basis $\{u_1, u_2, \ldots, u_k\}$ for $W_1 \cap W_2$. Extend it to a basis $\{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ for W_1 and to a basis $\{u_1, \ldots, u_k, w_1, \ldots, w_n\}$ for W_2. Then $\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\}$ generate. For this set to form a basis we need to check linear independence. We will argue by contradiction. Assume $a_1u_1 + \cdots a_ku_k + b_1v_1 \cdots + b_mv_m + c_1w_1 + \cdots + c_nw_n = 0$, with the constants not all zero. We must have at least some b_i nonzero, because otherwise we have a linear dependence relation among the u_i, w_j and they form a basis. Similarly a c_i must be nonzero. Then we have the relation $a_1u_1 + \cdots a_ku_k + b_1v_1 \cdots + b_mv_m = -c_1w_1 - \cdots - c_nw_n$. But the right hand side of the equation is an element in $W_2 \setminus W_1$ and the right hand side is an element in W_1. Contradiction.

Therefore, $\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\}$ is a basis for $W_1 + W_2$. So we have $\dim(W_1 + W_2 = k + m + n = (k+m) + (k+n) - k =$ $\dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

(b) Since $V = W_1 + W_2$, we only need to check that $W_1 \cap W_2 = \{0\}$. Since, by (a), $\dim(V) = \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$, the intersecion is the zero vector if and only if $\dim(W_1 \cap W_2)=0$, which is if and only if $\dim(V) = \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$.

1· 6.31 (a) Since $W_1 \cap W_2 \subset W_2$, we have $\dim(W_1 \cap W_2) \leq \dim(W_2) = n$.

(b) By [6.29] $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \leq \dim(W_1) + \dim(W_2) = m + n$

Chapter 2

2. 1.1 (a) True.

(b) False.

(c) True.

(d) True.

(e) False.

(f) False.

(g) True.

(h) False.

2.1.4 The null space has dimension 4 and a basis for the null space of T is:

$$\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The dimension of the range of T is 2 and a basis is:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Thus we have 4+2 = 6 = dim(V). T is neither one-to-one nor onto.

2.1.6 The dimension of the null space is $n^2 - 1$. Since the null space is exactly the subspace of trace zero matrices, by last week's homework we know $\{E_{i,j}, E_{1,1} - E_{k,k} : 1 < k \leq n, 1 \leq i, j \leq n, i \neq j\}$ is a basis.

The dimension of the range is 1 and a basis is 1_F.

Thus we have $n^2 - 1 + 1 = n^2 = dim M_{n \times n}(F)$. T is onto.

2.1.9 (d) $T(-1,0) = (1,0) \neq (-1,0) = -T(1,0)$

(e) $T(0,0) = (1,0 \neq (0,0) = 0_{R^2}$

2.1.10 $T(2,3) = T((3 \cdot (1,1) - (1,0)) = 3T(1,1) - T(1,0) = (6,15) - (1,4) = (5,11)$. T is one-to-one, because since the dim R(T) = 2, by the rank-nullity theorem, dim N(T) = 0.

2.1.12 No, there is no such T. By linearity $T(-2,0,6) = -2 \cdot T(1,0,3)$.