FRIDAY, APRIL 19, 2019

# LECTURE 9 - THE MATRIX OF A LINEAR TRANSFORMATION (SECTION 2.2)

:) HAPPY F, AND TODAY WE'RE GOING TO DO SOMETHING UNBELIEVABLE! WE'LL TAKE MATRICES (WHICH ARE JUST TABLE OF NUMBERS), AND BRING THEM TO LIFE ($\equiv$ FRANKENSTEIN OF MATH)

## I - COORDINATES

FOR THIS, WE'LL FIRST NEED TO DEFINE COORDINATES, WHICH IS A NEAT WAY OF ASSIGNING A "BARCODE" TO AN ABSTRACT VECTOR.

**FACT** (SECTION 1.6)

IF $\beta = \{v_1, \dots, v_n\}$ IS AN (ORDERED) BASIS FOR $V$ AND $x \in V$, THEN THERE ARE UNIQUE CONSTANTS $a_1, \dots, a_n \in \mathbb{F}$ SUCH THAT

$$x = a_1 v_1 + \dots + a_n v_n \qquad \text{(LET'S PUT THOSE CONSTANTS TOGETHER)}$$

**DEF** $[x]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ = COORDINATES OF $x$ WITH RESPECT TO $\beta$

**EX** $V = P_2$, $\beta = \{1, x, x^2\}$, $p = 2 + 3x + 4x^2 = \underline{2}\,1 + \underline{3}\,x + \underline{4}\,x^2$

$$[p]_\beta = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

**NOTE** ORDERED MEANS $\{1, x, x^2\} \neq \{x^2, x, 1\}$ (ORDER MATTERS)

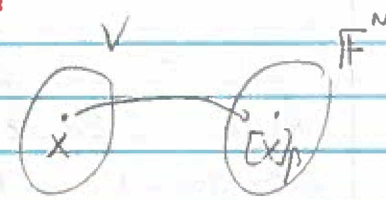
**EX** $V = \mathbb{R}^2$, $\beta = \{(1,1), (1,-1)\}$ $\quad x = (2,4) = \underline{3}\,(1,1) + (\underline{-1})(1,-1)$

$$[x]_\beta = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

**FACT** $T: V \to \mathbb{F}^N$, $T(x) = [x]_\beta$ IS LINEAR. MORE PRECISELY:

1) $[x+y]_\beta = [x]_\beta + [y]_\beta$

2) $[cx]_\beta = c[x]_\beta$

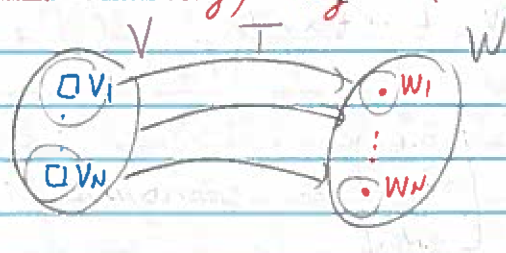



(WE'LL NEED THAT FOR TODAY)

(YOU MIGHT THINK: WAIT A MOMENT, DOES EVERY VS THEN JUST BECOME $\mathbb{F}^N$? INDEED! AND WE'LL SEE THAT IN SECTION 2.4)

## II - MATRICES OF LINEAR TRANSFORMATIONS

**RECALL** **FACT** IF $\beta = \{v_1, \dots, v_N\}$ IS A BASIS OF $V$ AND $w_1, \dots, w_N$ ARE ANY VECTORS IN $W$ THEN THERE IS A UNIQUE LT $T: V \to W$ SUCH THAT $T(v_j) = w_j$ FOR ALL $j$




(SO TO DEFINE A LT, YOU JUST NEED TO GIVE ME $T$ AT THE BASIS VECTORS. THIS BEGS THE Q: WHY NOT JUST PUT ALL THE $T(v_i)$ TOGETHER IN A TABLE? THIS IDEA ALMOST WORKS!)

**EX** LET $T: P_3 \to P_2$, $T(p) = p'$

FIND THE MATRIX $A$ OF $T$ WITH RESPECT TO THE BASES

$$\beta = \{\overset{v_1}{1}, \overset{v_2}{x}, \overset{v_3}{x^2}, \overset{v_4}{x^3}\} \text{ OF } V = P_3$$

AND $\gamma = \{1, x, x^2\}$ OF $W = P_2$

$T(v_1) = T(1) = (1)' = 0$

$T(x) = (x)' = 1$

$T(x^2) = 2x$

$T(x^3) = 3x^2$

GUESS $A = \begin{bmatrix} 0 & 1 & 2x & 3x^2 \end{bmatrix}$ AWKWARD!

(IDEALLY, WOULD LIKE MATRICES TO HAVE NUMBERS! MAN, IF ONLY WE COULD ATTACH A LIST OF NUMBERS TO OUR VECTORS!)

SOLUTION USE COORDINATES!

$[T(1)]_\gamma = [0]_\gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad 0 = \underline{\underline{0}}\,1 + \underline{\underline{0}}\,x + \underline{\underline{0}}\,x^2$

$[T(x)]_\gamma = [1]_\gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$[T(x^2)]_\gamma = [2x]_\gamma = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$[T(x^3)]_\gamma = [3x^2]_\gamma = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

ANS $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ (DERIVATIVE IN A BOX!)

SUMMARY

DEF IF $T: V \to W$ IS LINEAR,
$\beta = \{v_1, \dots, v_n\}$ IS A BASIS FOR $V$ ($N \sim$ INPUT)
$\gamma = \{w_1, \dots, w_m\}$ IS A BASIS FOR $W$ ($M \sim$ MOUTHPUT)
THEN THE MATRIX $A$ OF $T$ IS THE MATRIX WHOSE $j^{TH}$ COLUMN IS $[T(v_j)]_\gamma$

NOTATION $A = [T]_\beta^\gamma \uparrow$ (READ FROM DOWN TO UP)

$$[T(v_j)]_\gamma$$

$$A = \begin{array}{c} w_1 \\ w_2 \\ \vdots \\ w_m \end{array} \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{ij} \\ a_{mj} \end{bmatrix}$$

$$T(v_j)$$

MORE PRECISELY IF $T(v_j) = a_{1j} w_1 + \cdots + a_{mj} w_m$

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

THEN $a_{ij}$ IS THE $(i,j)^{TH}$ ENTRY OF $A$

EX $T : \mathbb{R}^2 \to \mathbb{R}^3$ $\beta = \{\overset{v_1}{(1,0)}, \overset{v_2}{(0,1)}\}$, $\gamma = \{\overset{w_1}{(1,0,0)}, \overset{w_2}{(0,1,0)}, \overset{w_3}{(0,0,1)}\}$

$T(a_1, a_2) = (a_1 + 3a_2, a_1, 2a_1 - 4a_2)$

$j=2$ ~~Ex~~ $T(v_2) = T(0,1) = (3,0,-4) = \overset{a_{12}}{3} w_1 + \overset{a_{22}}{0} w_2 + \overset{a_{32}}{(-4)} w_3$

SO THE SECOND COLUMN OF $A$ IS $\begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array} \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$
$T(v_2)$

(SO NOW WE'VE SEEN HOW TO ATTACH A MATRIX TO A LT, AND HERE'S A NEAT FACT: GIVEN A MATRIX, I CAN TELL YOU EXACTLY WHAT THE LT IS)

FACT IF $[U]_\beta^\gamma = [T]_\beta^\gamma$, THEN $U = T$

WHY? BY DEF, FOR ALL $j$, $[U(v_j)]_\gamma = [T(v_j)]_\gamma$

SO $U(v_j) = T(v_j)$

SO $U = T$ (LAST TIME) ∎

## III – Matrix Algebra

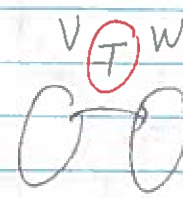
(Now that we've seen that LT are awesome, we may ask: what can we do to them? Today, we'll see that we can add them, 2.3: Multiplication, 2.4: Division)

**DEF** If $T \& U : V \to W$ are LT, then

1) $T+U : V \to W$ is defined by $(T+U)(x) = T(x) + U(x)$
2) $cT : V \to W$ " $(cT)(x) = cT(x)$

**Fact** $T+U$ and $cT$ are linear

But wait, there's more!




**DEF** $\mathcal{L}(V,W)$ = set of all LT from $V$ to $W$

**Fact** $\mathcal{L}(V,W)$ is itself a VS! (with $+$ and $\cdot$ as above)

**Note** $\underline{0} = T_0$ (zero transf)

And indeed, we can show that LT addition respects matrix addition

**Fact** If $T, U : V \to W$ are linear, then

1) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
2) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$

**Why?** 1) The $j^{th}$ column of $[T+U]_\beta^\gamma$ is

$$[(T+U)(v_j)]_\gamma = [T(v_j) + U(v_j)]_\gamma$$

$$= [T(v_j)]_\gamma + [U(v_j)]_\gamma$$

$$= j^{th} \text{ col of } T + j^{th} \text{ col of } U \checkmark$$

EX IF $[T]_\beta^\gamma = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $[U]_\beta^\gamma = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$,

THEN $[T+U]_\beta^\gamma = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 7 & 6 \end{bmatrix}$